Parametric Study of a Single Flexible Link Based on Timoshenko and Euler-Bernoulli Beam Theories

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Abstract

In this research the sensitivity analysis (SA) of the geometric parameters such as: length, thickness and width of a single link flexible manipulator on maximum deflection (MD) of the end effector and vibration energy (VE) of that point are conducted. The motion equation of the system is developed based on Gibbs-Appel (G-A) formulation. Also for modeling the elastic property of the system the assumption of assumed modes method (AMM) is applied. In this study, two theories are used to obtain the end-point MD and VE of the end effector. Firstly, the assumption of Timoshenko beam theory (TBT) has been applied to consider the effects of shear and rotational inertia. After that, Euler-Bernoulli beam theory (EBBT) is used. Then Sobol’s sensitivity analysis method is applied to determine how VE and end-point MD is influenced by those geometric parameters. At the end of the research, results of two mentioned theories are compared.

Keywords: Euler-Bernoulli, Timoshenko, Sensitivity analysis, Sobol.

Introduction

Robotic manipulators are extensively used in industries. Most of them are built in a rigid body to minimize the vibration of the end-effector to achieve the acceptable accuracy, by using the heavy materials and massive design. Since, the rigid manipulators are inefficient in terms of power consumption or speed; it is very desirable to build the robotic manipulators in a flexible body to reduce the weight of the arms to increase their speed of operation. Due to this, understanding and analyzing of flexible manipulations has concerned researchers for many years (Hermle et al., 2005 and Kumar et al., 2005), and proper modeling helps to the understanding of the process.

SA is fundamental tool in the designing, building, use and understanding of mathematical models of all forms (Taranatola et al., 2003). SA may be used here to determine the sensitivity amount of the factors that mostly take apart in the output variability (Xu et al., 2007; Kucherenko et al., 2008). SA has been used widely in other sciences to analyze models (Ellwein et al., 2008; Korayem et al., 2009; korayem et al., 2010), but this type of analysis has not been used extensively to our knowledge for the analysis of the elastic manipulators. The SA results will be very useful for selecting the appropriate link to achieve the optimum design by adjusting the dimensions of the flexible link. So, it is reasonable to use SA as a reliable tool to specify the effect of each parameter while the others are also changing.

Generally SA is classified in two groups (Saltelli et al., 2000): local SA and global SA. The global SA is used to study the effect of random input variables on the response variability of a computer code (Jourdan et al., 2012). Currently in most of studies, global SA techniques are used instead of local SA (Confalonieri et al., 2010; Saltelli et al., 2008; Taranatola et al., 2012). A commonly used method in global SA is the method of Sobol’, (Sobol’ et al., 1993; Saltelli et al., 1995; Saltelli et al., 1999). The Sobol indices (Korayem et al., 2009) are commonly used to distinguish the contribution of each input variables in the response variance decomposition. Moreover, dynamics of a single link flexible manipulator is simulated and sensitivity analysis of all geometric parameters of a dynamic model during the manipulation have been developed to obtain how each parameter is influenced on vibrations and deflections of the end-effector. In this case dynamic model of
Kinematics of the Single-Link Elastic Robotic Manipulator

Figure 1, shows the single-link elastic manipulator system considered in this research, where XYZ and xyz represent the stationary and moving coordinate’s frames, respectively.

The position vector of differential element \( \mathbf{Q} \) with respect to the base reference system \( \text{xyz} \) is shown by \( \tilde{\mathbf{r}}_{\text{Q}_0} \). Using modal analysis approach the deflection of the link is incorporated so, \( \tilde{\mathbf{r}}_{\text{Q}_0} \) is expressed as,

\[
\tilde{\mathbf{r}}_{\text{Q}_0} = \tilde{\mathbf{r}} + \{u \ v \ w\}^T \tag{1}
\]

Where \( \tilde{\mathbf{r}} = [\eta \quad 0 \quad 0]^T \) is the position vector of differential element \( \mathbf{Q} \) with respect to \( \mathbf{o} \), until the flexible link is undeformed; and \( u \), \( v \) and \( w \) are small displacements along the \( \mathbf{o}_x \), \( \mathbf{o}_y \) and \( \mathbf{o}_z \) axes, respectively. The assumed mode method approach is used to determine these small displacements as,

\[
\{u \ v \ w\}^T = \sum_{i=1}^{m} \delta_i(t) \tilde{\mathbf{r}}_i(\eta) \tag{2}
\]

Where \( \tilde{\mathbf{r}}_i = [x_i \ y_i \ z_i]^T \) is the Eigen function vector. The components \( x_i \), \( y_i \) and \( z_i \) are \( i \)-th longitudinal and transverse mode shapes of the link; \( \delta_i \) is the \( i \)-th time dependent modal generalized coordinate of the link; and \( m \) is the number of modes used to express the deflection of the link. The centerline’s total transverse displacement of differential element \( \mathbf{Q} \) is due to bending and shear. So the total slopes of the deflected centerline about \( \mathbf{o}_y \) and \( \mathbf{o}_z \) axes due to the bending and shear deformation can be represented as,

\[
-\frac{\partial w}{\partial \eta} = \varphi_y + \theta_y \tag{3}
\]

\[
\frac{\partial v}{\partial \eta} = \varphi_z + \theta_z \tag{4}
\]

Where \( \varphi_y \) and \( \varphi_z \) are the slope of the deflected centerline due to shear and \( \theta_y \), \( \theta_z \) are the slope of the deflected centerline due to bending. Since the shear has not any effects on rotating the differential element \( \mathbf{Q} \), so this differential element undertakes rotations only due to bending and torsion. Hence the
rotation of this element around the ox, oy and oz axes can be considered as \( \theta_x, \theta_y \) and \( \theta_z \), respectively. The truncated modal expansion of those mentioned small angles can be represented as,

\[
\tilde{\theta} = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \sum_{i=1}^{m} \delta_i(t) \tilde{\theta}_i(\eta) \tag{5}
\]

Where \( \tilde{\theta}_i = \begin{bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{bmatrix}^T \) is the Eigen function vector whose components \( \theta_{xi}, \theta_{yi} \) and \( \theta_{zi} \) are i-th rotational mode shapes of the link about \( ox, oy \) and \( oz \) axes, respectively.

**The System Gibbs Function and Its Derivatives**

In this section the acceleration energy of the system and its derivatives are developed to construct the G-A formulation with respect to quasi-accelerations. Considering the assumption of TBT, the acceleration energy of a differential element \( Q \) can be represented as follow:

\[
ds = \frac{1}{2} \mu(\eta) \left( \ddot{\mathbf{r}}_Q^T \cdot \ddot{\mathbf{r}}_Q \right) d\eta + \frac{1}{2} \ddot{\mathbf{\omega}}^T \cdot J(\eta) \ddot{\mathbf{\omega}} d\eta \tag{6}
\]

Though, with the assumption of EBBT only the first term of Equation (6) should be preserved. Also \( \mu(\eta) \) and \( J(\eta) \) are mass per unit length and mass moment of inertia per unit length, respectively. \( \dddot{\mathbf{r}}_Q \) and \( \dddot{\mathbf{\omega}} \) are linear and angular acceleration of differential element \( Q \) that can be presented as:

\[
\dddot{\mathbf{r}}_Q = \dddot{\mathbf{r}}_{Qo} + 2\dddot{\mathbf{\omega}} \times \dddot{\mathbf{r}}_{Qo} + \dddot{\mathbf{\omega}} \times \left( \dddot{\mathbf{\omega}} \times \dddot{\mathbf{r}}_{Qo} \right) \tag{7}
\]

\[
\dddot{\mathbf{\omega}} = \sum_{i=1}^{m} \dddot{\delta}_i(t) \tilde{\theta}_i(\eta) \tag{8}
\]

Note that, in above expressions, \( \dddot{\mathbf{\omega}} \) and \( \dddot{\mathbf{r}} \) are angular velocity and angular acceleration of the link, respectively. Also the velocity and the acceleration of differential element \( Q \) with respect to the origin of the local reference system are shown by \( \dddot{\mathbf{r}}_{Qo} \) and \( \dddot{\mathbf{\omega}}_{Qo} \), respectively. Inserting Equation (7) and Equation (8) into Equation (6) and integrating over the link from 0 to 1, the total acceleration energy of the link will be obtained as:

\[
S = \frac{1}{2} B_1 - 2\dddot{\mathbf{\omega}}^T \cdot \dddot{B}_2 + \dddot{\mathbf{\omega}}^T \cdot \dddot{B}_3 - \dddot{\mathbf{\omega}}^T \cdot B_4 \dddot{\mathbf{\omega}} + 2\dddot{\mathbf{\omega}} \cdot B_5 \dddot{\mathbf{\omega}} + \frac{1}{2} \dddot{\mathbf{\omega}}^T \cdot B_6 \dddot{\mathbf{\omega}} + \frac{1}{2} B_7\tag{9}
\]

\[
+ \text{irrelevant terms}
\]

Where \( \dddot{\mathbf{\omega}} \) is the skew-symmetric tensor associated with \( \dddot{\mathbf{\omega}} \) vector. Also there is a term named as “irrelevant terms”. Taking the partial derivatives of Gibbs’ function with respect to quasi-accelerations, motion equation with G-A formulation will be constructed. So, all the terms in Gibbs’ function which do not contain \( \dddot{\mathbf{r}} \) and \( \dddot{\mathbf{\omega}} \) can be ignored. The variables appeared in Equation (9) can be computed as:

\[
B_1 = \sum_{j=1}^{m} \sum_{k=1}^{m} \dddot{\delta}_j \tilde{C}_{jk} \\
B_2 = \sum_{j=1}^{m} \sum_{k=1}^{m} \dddot{\delta}_j \tilde{C}_{jk} \tilde{\mathbf{C}}_{jk} \\
B_3 = \sum_{j=1}^{m} \dddot{\delta}_j \tilde{\mathbf{C}}_{jk} \\
B_4 = \sum_{j=1}^{m} \dddot{\delta}_j \beta_j \\
B_5 = \sum_{j=1}^{m} \dddot{\delta}_j \beta_j \\
B_6 = C_3 + \sum_{j=1}^{m} \dddot{\delta}_j \left( C_{6j}^T + \beta_j \right) \\
B_7 = C_3 + \sum_{j=1}^{m} \dddot{\delta}_j \left( C_{7j}^T + \beta_j \right) \\
\text{where} \\
C_{1jk} = \int_0^1 \mu \dddot{\mathbf{r}}_j^T \cdot \dddot{\mathbf{r}}_k d\eta \\
\tilde{C}_{2jk} = \int_0^1 \mu \dddot{\mathbf{r}}_j \dddot{\mathbf{r}}_k d\eta \\
C_{3j} = \int_0^1 \mu \dddot{\mathbf{r}}_j^T \dddot{\mathbf{r}}_j d\eta \\
C_{4j} = \int_0^1 \mu \dddot{\mathbf{r}}_j^T \dddot{\mathbf{r}}_j d\eta \\
C_{5jk} = \int_0^1 \mu \dddot{\mathbf{r}}_j^T \dddot{\mathbf{r}}_k d\eta \\
\tilde{C}_{6j} = \int_0^1 \mu \dddot{\mathbf{r}}_j \dddot{\mathbf{r}}_j d\eta \\
C_{7jk} = \int_0^1 \mu \dddot{\mathbf{r}}_j^T \dddot{\mathbf{r}}_k d\eta \\
\tilde{\alpha}_j = \tilde{C}_{6j} + \sum_{k=1}^{m} \dddot{\delta}_k \tilde{C}_{2jk} \\
\beta_j = C_{6j} + \sum_{k=1}^{m} \dddot{\delta}_k C_{7jk} \tag{17-25}
\]

Where \( \dddot{\mathbf{r}}_j \) and \( \dddot{\mathbf{\omega}}_j \) are skew-symmetric tensor associated with \( \dddot{\mathbf{r}}_j \) and \( \dddot{\mathbf{\omega}}_j \) vectors. As mentioned above, one part of dynamic equations of the system using G-A formulation will be obtained by differentiating of Gibbs’ function with respect to quasi-accelerations. So, these two terms can be represented as,
The System's Potential Energy and Its Derivatives

The system's potential energy arises from two sources, first potential energy due to gravity and second potential energy due to the elastic deformations. The potential energy due to the gravity can be considered simply by substituting \( g \), where \( g \) is the acceleration of gravity.

To represent the strain potential energy stored in flexible link, two theories are existed; TBT and EBBT. For the first assumption the strain potential energy will be represented in terms of deflections and rotations as,

\[
V_e = \frac{1}{2} \int \left[ kAG \left( \varphi_x^2 + \varphi_y^2 \right) + EI_x \left( \frac{\partial \theta_x}{\partial \eta} \right)^2 + EI_y \left( \frac{\partial \theta_y}{\partial \eta} \right)^2 + EA \left( \frac{\partial ^2 u}{\partial \eta^2} \right)^2 + GI_x \left( \frac{\partial \theta_x}{\partial \eta} \right)^2 \right] K_j \delta_j(t) \partial \delta_j \tag{28}
\]

But for the second assumption, the first term of the above integral will be eliminated. In Equation (28), \( E \) and \( G \) are the modulus of elasticity and shear modulus, respectively; \( I_x \) is the polar area moment of inertia about \( ox \) axis; \( I_y \) and \( I_z \) are the area moment of inertia about \( oy \) and \( oz \) axes, respectively; \( A \) is the cross section area of the link and \( k \) is shear correction factor. As noted in previous Section the small angles \( \theta_x, \theta_y, \theta_z \) and small displacements \( u, v, w \) can be represented with a truncated modal approximation.

By substituting these expressions in Equation (28) the strain potential energy of the link will be obtained as,

\[
V_e = \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} \delta_j(t) \delta_k(t) K_{jk} \tag{29}
\]

where

\[
K_{jk} = \left[ kAG \left( \varphi_x \varphi_y + \varphi_y \varphi_z \right) + EI_x \frac{\partial \theta_x}{\partial \eta} \frac{\partial \theta_y}{\partial \eta} + EI_y \frac{\partial \theta_y}{\partial \eta} \frac{\partial \theta_x}{\partial \eta} + EA \frac{\partial u}{\partial \eta} \frac{\partial u}{\partial \eta} + GI_x \frac{\partial \theta_x}{\partial \eta} \frac{\partial \theta_x}{\partial \eta} \right] \tag{30}
\]

To derive the motion equation of the elastic robotic manipulators, the partial derivatives of strain potential energy with respect to generalized coordinates are needed. So, these two terms can be represented as,

\[
\frac{\partial V_e}{\partial \delta_j} = 0 \tag{31}
\]

\[
\frac{\partial V_e}{\partial \delta_j} = \sum_{k=1}^{m} \delta_k(t) K_{kj} \tag{32}
\]

Dynamic Equations of Flexible Link Manipulator

Motion equation of elastic robotic manipulators will be completed by considering the generalized forces which are caused by the remaining external force terms. Let us assume that there is no external load on the links of the considered robotic manipulator. So, the generalized forces in the deflection equations will be zero. The generalized force in the joint equations is the torque \( \tau \) that applies to the joint. With this assumption, the dynamic equations of motion in the G-A formulation will be completed as follows:

- The joint equations of motion
  \[
  \frac{\partial S}{\partial \delta_j} + \frac{\partial V_e}{\partial \delta_j} = \tau \tag{33}
  \]

- The deflection equations of motion
  \[
  \frac{\partial S}{\partial \delta_j} + \frac{\partial V_e}{\partial \delta_j} = 0 \quad j = 1, 2, \ldots, m \tag{34}
  \]

Sobol’s Sensitivity Analysis Method

Sobol’s sensitivity analysis is one of the well-known statistical methods that is used successfully to non-linear mathematical models. (Gloda et al., 2008) showed that, it can be used efficiently for model based analysis of
real-world rough-terrain robotic systems. To explain Sobol’s method, the input factors region should be defined as follows:

\[ \Omega^k = (X|0 \leq x_i \leq 1; i = 1, 2, \ldots, k) \] (35)

Where \( x_i \) is input factors vector, that are perpendicular to each other’s. The main idea behind the Sobol’s method is that, the function \( f(x) \) is derived from the sum of the following functions;

\[
f(x_1, x_2, \ldots, x_k) = f_0 + \sum_{i=1}^{k} f_i(x_i) + \sum_{i \leq j \leq k} f_{ij}(x_i, x_j) + \ldots + f_{1,2,\ldots,k}(x_1, x_2, \ldots, x_k)
\] (36)

The first term of the above equation is determined as,

\[ f_0 = \int f(x)dx \] (37)

Sobol showed that, the decomposition Equation (36) is unique and that all the terms can be computed via multidimensional integrals,

\[
f_i(x_i) = -f_0 + \int \int \ldots \int f(x)dx
\] (38)

\[
f_{ij}(x_i, x_j) = -f_0 - \sum f_i(x_i) + \int \int \ldots \int f(x)dx_{-ij}
\] (39)

where \( dx_{-ij}, dx_{-ij} \) represent integration over all variables except \( x_i \) and \( x_j \), respectively. Hence, for higher-order terms, continuous formula can be obtained. In the sensitivity indices which is based on variance, total variance of \( f(x) \) “D” is expressed to be:

\[ D = \int f^2(x)dx - (f_0)^2 \] (40)

After squaring and integrating Equation (36) over all variables, expression \( D \) is simplified as follow:

\[ D = \sum_{i=1}^{k} D_i + \sum_{i \leq j \leq k} D_{ij} + D_{1,2,\ldots,k} \] (41)

So the sensitivity measures \( S_{1,2,\ldots,k} \), are given by:

\[ S_{1,2,\ldots,k} = \frac{D_{1,2,\ldots,k}}{D} \quad 1 \leq i_1 \leq \ldots \leq i_s \leq k \] (42)

The total sensitivity analysis index is obtained by adding all the sensitivity indices involving the factor in Equation (42). In the proposed method, Sobol’s sensitivity analysis is applied to evaluate the optimal value of dimensions of the flexible link with respect to the VE and end-effector’s MD minimization.

Results and Discussions

Sensitivity Analysis of VE Using TBT and EBBT

In this section both TBT and EBBT assumptions are used to achieve VE of the single link flexible manipulator for each application. The assumption of TBT has been applied to include the effects of shear and rotational inertia. To do this, the variation intervals of each parameter should be extracted. Table 1, presents those intervals. Then, Sobol’s sampling method is applied to generate 1152 uniform random numbers on intervals presented in Table 1. Using the approach discussed in the previous section the VE of the flexible link is obtained with respect to each extracted random number. The results of the SA of the flexible link with respect to TBT and EBBT are shown in Table 2. Also, the Pie chart diagrams of the SA of the VE of the elastic link using both TBT and EBBT are illustrated in Figures 2 and 3, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Specifications</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>Steel</td>
<td>-</td>
</tr>
<tr>
<td>Density (( \rho ))</td>
<td>7800</td>
<td>kg/m³</td>
</tr>
<tr>
<td>Young module (E)</td>
<td>200</td>
<td>Gpa</td>
</tr>
<tr>
<td>Length</td>
<td>(20, 140)</td>
<td>Cm</td>
</tr>
<tr>
<td>Thickness</td>
<td>(0.1, 0.2)</td>
<td>Cm</td>
</tr>
<tr>
<td>Width</td>
<td>(4, 7)</td>
<td>Cm</td>
</tr>
</tbody>
</table>

Table 1. Properties of the link
Table 2. The SA results of VE using TBT and EBBT.

<table>
<thead>
<tr>
<th>Sensitivity Indices Values</th>
<th>Values (TBT)</th>
<th>Values (EBBT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_L</td>
<td>0.0985</td>
<td>0.0983</td>
</tr>
<tr>
<td>S_T</td>
<td>0.1055</td>
<td>0.1060</td>
</tr>
<tr>
<td>S_W</td>
<td>0.0137</td>
<td>0.0140</td>
</tr>
<tr>
<td>S_LT</td>
<td>0.9580</td>
<td>0.9539</td>
</tr>
<tr>
<td>S_LW</td>
<td>0.77</td>
<td>0.7684</td>
</tr>
<tr>
<td>S_TW</td>
<td>0.9637</td>
<td>0.9625</td>
</tr>
</tbody>
</table>

Figure 2. Pie chart diagram of SA using TBT

Figure 3. Pie chart diagram of SA using EBBT

Figure 4. Effects of the length on the VE using TBT and EBBT

Figure 5. Effects of the thickness on the VE using TBT and EBBT

Figure 6. Effects of the width on the VE using TBT and EBBT

Figure 7. Effects of L/WT on the VE using TBT and EBBT

Sensitivity analysis of MD of the end-effector using TBT and EBBT

As the previous sub-section, the assumptions of TBT and EBBT are used here, too. MD of the end-effector of the elastic link correspond to those 1152 random numbers are computed. The SA results of TBT and EBBT assumptions are obtained by using the Sobol’s method and they are presented in Table 3. Also, the pie chart diagrams of the SA are illustrated in Figures 8 and 9.
Table 3. The SA results of MD using TBT and EBBT.

<table>
<thead>
<tr>
<th>Sensitivity Indices</th>
<th>Values (TBT)</th>
<th>Values (EBBT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_L</td>
<td>0.3236</td>
<td>0.3228</td>
</tr>
<tr>
<td>S_T</td>
<td>0.2173</td>
<td>0.2169</td>
</tr>
<tr>
<td>S_W</td>
<td>0.0167</td>
<td>0.0170</td>
</tr>
<tr>
<td>S_LT</td>
<td>1.0530</td>
<td>1.0531</td>
</tr>
<tr>
<td>S_LW</td>
<td>0.7587</td>
<td>0.7583</td>
</tr>
<tr>
<td>S_TW</td>
<td>0.6582</td>
<td>0.6584</td>
</tr>
</tbody>
</table>

According to the Figures 8 and 9, it is understood that, the SA results of the MD of the end-effector of the flexible link are the same with both TBT and EBBT assumptions. But like the previous sub-section, by studying the numerical results of the SA, it is concluded that the results are different. As shown in Figures 8 and 9, the most effective parameter among the first sensitivity indices, is S_L, which shows the sensitivity amount of length on the MD of the end-effector.

To discover how each parameter influenced on the MD of the end-effector, simulations are done. Figures 10, 11 and 12 show the results. Each figure shows the results corresponding to the both TBT and EBBT simultaneously.

According to the Figures 10, 11 and 12, the MD of the end-effector is increased by the growth of length. But increasing the amount of thickness and also width are led to decrease the MD. So, like the previous sub-section, to find the optimal values of dimensions to achieve the minimum MD, the factor of \( \frac{L}{WT} \) is defined too, here. Figure 13, shows the relation between MD and \( \frac{L}{WT} \), by using TBT and EBBT assumptions. According to the Fig.13, the best values of \( \frac{L}{WT} \) is 20 (\( \frac{1}{cm} \)). As noted, the dimensions of the flexible link should be selected so that the value of \( \frac{L}{WT} \) must be equal to 20 (\( \frac{1}{cm} \)).
Conclusion

In this research, dynamic modeling of the single link flexible manipulator is developed based on G-A formulations. AMM is applied to achieve the elastic modeling of the system. The behavior of the VE and MD of the end-effector are studied to obtain appropriate criteria for mechanical design of the system. Due to this, both TBT and EBBT assumptions are applied to achieve the VE and MD of the end-effector. Understanding the effects of each geometric parameter on VE and MD, SA is done by using Sobol's method. After studying the effects of each geometric parameter, the relation between those parameters vs VE and MD are presented. The results show that, to decrease the VE and MD, the flexible link with low length and high thickness and width should be selected. It is shown that, the most sensitive parameter corresponds to the length, either TBT or EBBT. Moreover, the most percentage of sensitivity among all the other sensitivity indices is corresponded to $S_{LT}$, which expresses the effects of length and thickness simultaneously.

According to the results, the optimum values of VE and MD occur at $50 \frac{cm}{LWT}$ and $20 \frac{cm}{LWT}$ for $L/WT$, respectively.

References