Heat Conduction of a Hollow Cylinder via Generalized Hankel Transform

A. M. Shafei, S. Rafee Nekoo*

Ph.D. Student of Mechanical Engineering, School of Mechanical Engineering, Iran University of Science and Technology, P.O. Box 1684613114, Tehran, Iran

*Correspondent Author, Email: rafee@iust.ac.ir

ABSTRACT: In this work, the heat conduction problem of a finite hollow cylinder is solved as an exact solution method. In order to solve the PDE equation, generalized finite Hankel and other integral transformations are used. Two case studies for simulations are presented and verified with simulations which extracted from finite element method. The results are shown that this approach is proper for solving heat conduction problems in cylindrical coordinate.

Keywords: Generalized Finite Hankel Transform; Hollow Cylinder; Finite Element; Heat Conduction

Introduction

Direct measurement of distribution of temperature is difficult and not possible some times. For dealing with this problem, different methods have been used. In (Sermet, 2012) a new Green’s function in closed form was introduced for a boundary value problem in thermoelastoestatics for a quadrant domain. Shiah and Lee (2011) expressed the problem of 3D modeling of anisotropic heat conduction problem with arbitrary volume heat source. In order for solving the problem, a Boundary Integral Equation and Multiple Reciprocity Method were applied and three numerical examples were explained to demonstrate the accuracy of the method. Baldasseroni et al. (2011) demonstrated heat transfer simulation accompanied by thermal measurements of microfabricated x-ray transparent heater stages. An amorphous silicon nitride membrane was introduced appropriate for the sample. Thermal specifications in air and vacuum were analyzed, analytically and with simulation. Hoshan (2009) presented a triple integral equation method for solving heat conduction equation. A new kind of triple integral was employed to find a solution of non-stationary heat equation in an axisymmetric cylindrical coordinates under mixed boundary of the first and second kind conditions, using Laplace transform the triple integral turned into a singular integral of the second type. Cossali (2009) expressed an analytical solution of the steady periodic heat conduction in a solid homogenous finite cylinder via Fourier transform, with the sole restriction of uniformity on the lateral surface and radial symmetry on the bases. A harmonic heating, as an example, introduced accompanied with simulation results. In (Lambrakos et al., 2008) inverse analysis of heat conduction problem in hollow cylinder with axisymmetric source distribution was explained with experimental results. Golytsina (2008) presented mathematical simulation of temperature field in a hollow rotating cylinder with nonlinear boundary conditions. Volle et al. (2008) introduced a semi-analytical method for inverse heat conduction of a rotating cylinder using integral transform methods. In this paper, in order to solve the problem as an exact method, generalized finite Hankel transform is used. Hankel transform is used for solving problems consist of cylindrical coordinates, but not the hollow cylinders. Eldabe et al (2004) introduced an extension of the finite Hankel transform, which is capable of solving hollow cylindrical coordinates, heat equation or wave with mixed boundary values. Povstenko (2008) expressed the radial diffusion in a cylinder via Laplace and Hankel transform. Akhtar (2001) presented exact solutions for rotational flow of a generalized Maxwell fluid between two circular cylinders.
Modeling of the Axis Symmetric Problem

In this section, the distribution of temperature in a hollow cylinder is considered, as it is shown in figure (1). The equation of the problem is in the form of equation (1):

\[
\begin{align*}
\frac{c^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + u_{zz} &= u_t,
\end{align*}
\]

(1)

Figure (1): Schematic of the hollow cylinder.

Since the case is not dependent on \( \theta \), equation (1) can be written as equation (2):

\[
\begin{align*}
\frac{c^2}{r} \frac{\partial^2 u}{\partial r^2} + u_{zz} &= u_t,
\end{align*}
\]

(2)

In which:

\[
\begin{align*}
u_{rr} &= \frac{\partial^2 u(r, z, t)}{\partial r^2} \\
u_r &= \frac{\partial u(r, z, t)}{\partial r} \\
u_{zz} &= \frac{\partial^2 u(r, z, t)}{\partial z^2} \\
u_t &= \frac{\partial u(r, z, t)}{\partial t}
\end{align*}
\]

Where \( u \) is the temperature, \( Z \) the height, \( r \) the radius of the cylinder. Also \( t \) stands for the time and \( c \) for diffusivity factor. The boundary and initial condition are considered in equations (3-7):

\[
\begin{align*}
u(a, z, t) &= 0 \\
u(b, z, t) &= T_b e^\gamma \\
u(r, 0, t) &= 0 \\
u(r, h, t) &= 0
\end{align*}
\]

(3-7)
$u(r, z, t) = 0$ \hspace{1cm} (7)

Where $T_0$ is a constant temperature. In order to solve the PDE equation, integral transformations are used. The first transformation is Laplace which changes the domain of time to $s$, equation (8):

$$c^2 \left( \overline{u}_n + \frac{1}{r} \overline{u}_r + \overline{u}_{zz} \right) = s \overline{u} - u(r, z, 0)$$ \hspace{1cm} (8)

In which:

$$\overline{u}(r, z, s) = \mathcal{L}\{u(r, z, t)\}$$

$$\overline{u}_n = \frac{\partial^2 u(r, z, s)}{\partial r^2}$$

$$\overline{u}_r = \frac{\partial u(r, z, s)}{\partial r}$$

$$\overline{u}_{zz} = \frac{\partial^2 u(r, z, s)}{\partial z^2}$$

Second transformation is Fourier Cosine and therefore equation (8) changes to equation (9):

$$c^2 \left( \overline{u}_n + \frac{1}{r} \overline{u}_r - n^2 \pi^2 \overline{u}(r, n, s) - 2\left[ \overline{u}_r (r, 0, s) + (-1)^{n+1} \overline{u}_r (r, h, s) \right] \right) = s \overline{u}(r, n, s)$$ \hspace{1cm} (9)

Where:

$$\overline{u}(r, n, s) = \frac{1}{h} \int_0^h \overline{u}(r, z, s) \cos\left( \frac{n \pi}{h} z \right) dz$$

$$\overline{u}_n = \frac{\partial^2 u(r, n, s)}{\partial r^2}$$

$$\overline{u}_r = \frac{\partial u(r, n, s)}{\partial r}$$

The boundary conditions of the bottom and top of the cylinder appears in equation (9) which are zeroes. Forming the equations as following:

$$c^2 \left( \overline{u}_n + \frac{1}{r} \overline{u}_r \right) = (s + (cn)^2) \overline{u}(r, n, s)$$ \hspace{1cm} (10)

Finite Hankel transform can be applied. The general formulation is presented in equation (11) [9]:

$$H_m = \left[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \left( \frac{1}{x} - \frac{m^2}{x^2} \right) \right] = \frac{2}{\pi b^2} f\left( b \right) - \frac{2J_m(k_i b)}{\pi a^2 J_m(k_i a)} f\left( a \right) - k_i^2 \overline{f}(n)$$ \hspace{1cm} (11)

Selecting $l = m$ equal to zero in equation (11), the left side of equation (10) is appeared and employing the transformation results the equation (12):

$$c^2 \left( \frac{2}{\pi} \overline{u}(b, n, s) - \frac{2J_0(k_i b)}{\pi J_0(k_i a)} \overline{u}(a, n, s) - k_i^2 \hat{u}(k_i, n, s) \right) = (s + (cn)^2) \hat{u}$$ \hspace{1cm} (12)

In which:

$$\hat{u}(k_i, n, s) = H_0 \{ \overline{u}(r, n, s) \}$$

Simplifying and applying the radial boundary conditions equation (12) turn into equation (13):

$$\hat{u}(k_i, n, s) = \frac{4c^2}{\pi} \frac{1}{s + \left( (cn)^2 + (ck_i)^2 \right) s} T_0 \left( \frac{-1 + e(-1)^n}{1 + n^2 \pi^2} \right)$$ \hspace{1cm} (13)

We separate equation (13) into two parts $n = 0$ and $n > 0$. First, for $n > 0$, inverse finite Hankel Transform is applied in equation (14):
\[ \bar{u}(r, s, z) = \sum_{n=1}^{\infty} \frac{\pi^2 k^2 J^2_0(ak_r)}{2 J^2_0(ak_i) - J^2_0(bk_i)} \frac{4c^2}{s + \left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\}} \times \frac{T_b}{s} \left( -1 + e^{-(-1)^n} \right) \left[ J_0(k_r)Y_0(k, b) - J_0(k, b)Y_0(k, r) \right] \] (14)

Taking inverse Fourier Cosine transform, equation (15) is formed:

\[ u(r, s, z) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{\pi^2 k^2 J^2_0(ak_r)}{2 J^2_0(ak_i) - J^2_0(bk_i)} \frac{4c^2}{s + \left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\}} \times \frac{T_b}{s} \left( -1 + e^{-(-1)^n} \right) \left[ J_0(k_r)Y_0(k, b) - J_0(k, b)Y_0(k, r) \right] \cos \left( \frac{\alpha n}{h} z \right) \] (15)

Finally, inverse Laplace transform should be taken, equation (16):

\[ u(t, r, s) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} 2c^2 \pi T_b \frac{k^2 J^2_0(ak_r)}{2 J^2_0(ak_i) - J^2_0(bk_i)} \frac{1}{\left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\}} \times \left( -1 + e^{-(-1)^n} \right) \left[ J_0(k_r)Y_0(k, b) - J_0(k, b)Y_0(k, r) \right] \cos \left( \frac{\alpha n}{h} z \right) \times \left( 1 - e^{-\left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\} t} \right) \] (16)

In order to solve in steady state solution, \( t \) is considered infinity, so equation (16) is written in the form of equation (17):

\[ u(r, s) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} 2c^2 \pi T_b \frac{k^2 J^2_0(ak_r)}{2 J^2_0(ak_i) - J^2_0(bk_i)} \frac{1}{\left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\}} \times \left( -1 + e^{-(-1)^n} \right) \left[ J_0(k_r)Y_0(k, b) - J_0(k, b)Y_0(k, r) \right] \cos \left( \frac{\alpha n}{h} z \right) \times \left( 1 - e^{-\left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\} t} \right) \] (17)

The second part, \( n = 0 \) results equation (18):

\[ \frac{\hat{u}(k, 0, s)}{2} = \frac{2c^2}{\pi} \frac{1}{s + (ck_r)^2} \frac{T_b}{s} \left( e - 1 \right) \] (18)

In addition, taking inverse Finite Hankel and Laplace transforms change equation (18) to (19):

\[ u(t, r, s) = \sum_{n=1}^{\infty} \pi T_b \frac{k^2 J^2_0(ak_r)}{2 J^2_0(ak_i) - J^2_0(bk_i)} \frac{e - 1}{(ck_r)^2} \left[ J_0(k_r)Y_0(k, b) - J_0(k, b)Y_0(k, r) \right] \times \left( 1 - e^{-\left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\} t} \right) \] (19)

As steady-state analytical solution, equation (20) is appeared:

\[ u(t, r) = \sum_{n=1}^{\infty} \pi T_b \frac{J^2_0(ak_r)}{J^2_0(ak_i) - J^2_0(bk_i)} \left( e - 1 \right) \left[ J_0(k_r)Y_0(k, b) - J_0(k, b)Y_0(k, r) \right] \] (20)

The final steady-state solution is both parts of answers in equation (21):

\[ u(r, z) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} 2c^2 \pi T_b \frac{k^2 J^2_0(ak_r)}{2 J^2_0(ak_i) - J^2_0(bk_i)} \frac{1}{\left\{ \left( \alpha n \right)^2 + (ck_r)^2 \right\}} \times \left( -1 + e^{-(-1)^n} \right) \left[ J_0(k_r)Y_0(k, b) - J_0(k, b)Y_0(k, r) \right] \cos \left( \frac{\alpha n}{h} z \right) \] (21)
In order to simulate \( k_i \) should found by equation (22) [9]:

\[
J_0(k,b)Y_0(k,a) - J_0(k,a)Y_0(k,b) = 0
\]  

(22)

In which \( a = 0.5 \), \( b = 1 \). In table (1) the roots of equation (22) are presented.

| \( k_1 \) | 6.24 | \( k_{11} \) | 69.11 | \( k_{21} \) | 131.94 | \( k_{31} \) | 194.77 | \( k_{41} \) | 257.60 |
| \( k_2 \) | 12.54 | \( k_{12} \) | 75.39 | \( k_{22} \) | 138.22 | \( k_{32} \) | 201.06 | \( k_{42} \) | 263.89 |
| \( k_3 \) | 18.83 | \( k_{13} \) | 81.67 | \( k_{23} \) | 144.51 | \( k_{33} \) | 207.34 | \( k_{43} \) | 270.17 |
| \( k_4 \) | 25.12 | \( k_{14} \) | 87.96 | \( k_{24} \) | 150.79 | \( k_{34} \) | 213.62 | \( k_{44} \) | 276.45 |
| \( k_5 \) | 31.40 | \( k_{15} \) | 94.24 | \( k_{25} \) | 157.07 | \( k_{35} \) | 219.91 | \( k_{45} \) | 282.74 |
| \( k_6 \) | 37.69 | \( k_{16} \) | 100.52 | \( k_{26} \) | 163.36 | \( k_{36} \) | 226.19 | \( k_{46} \) | 289.02 |
| \( k_7 \) | 43.97 | \( k_{17} \) | 106.81 | \( k_{27} \) | 169.64 | \( k_{37} \) | 232.47 | \( k_{47} \) | 295.30 |
| \( k_8 \) | 50.26 | \( k_{18} \) | 113.09 | \( k_{28} \) | 175.92 | \( k_{38} \) | 238.75 | \( k_{48} \) | 301.59 |
| \( k_9 \) | 56.54 | \( k_{19} \) | 119.37 | \( k_{29} \) | 182.21 | \( k_{39} \) | 245.04 | \( k_{49} \) | 307.87 |
| \( k_{10} \) | 62.82 | \( k_{20} \) | 125.66 | \( k_{30} \) | 188.49 | \( k_{40} \) | 251.32 | \( k_{50} \) | 314.15 |

Considering \( T_b = 100^\circ C \), \( n = 10 \), \( c = 1 \), and \( i = 50 \) at \( z = 0.5 \) the distribution of temperature is shown in figure (2). For a better presentation \( \frac{3}{4} \) of the contour is displayed.

![Figure (2): The distribution of temperature at \( z = 0.5 \).](image)

In order to verify the results, Finite Element Method is used and the temperature field is demonstrated in figure (3).
The details of temperature field at $z = 0.5$ is shown in figure (4) and at $z = 1$ in figure (5).

Figure (4): The distribution of temperature at $z = 0.5$, Exact and FEM.
Figure (5): The distribution of temperature at \( z = 1 \), Exact and FEM.

Finally at \( r = 0.75 \) and \( 0 < z < 1 \) the comparison between methods is shown in figure (6).

Figure (6): The distribution of temperature at \( r = 0.75 \) and \( 0 < z < 1 \).

**Modeling of the General Problem**

In previous section the distribution of temperature was solved as an axis symmetric case and because the problem was independent of \( \theta \), generalized finite Hankel transform of zero order was used. In this part the problem and boundary conditions are dependent on \( \theta \), so the PDE equation (1) is used and boundary conditions are chosen as equations (23-27):

\[
\begin{align*}
  &u(a, 0, z, t) = 0 \\
  &u(b, 0, z, t) = T_b \theta e^z \\
  &u(r, 0, 0, t) = 0 \\
  &u(r, 0, h, t) = 0 \\
  &u(r, 0, z, 0) = 0 
\end{align*}
\]  

(23) \hspace{2cm} (24) \hspace{2cm} (25) \hspace{2cm} (26) \hspace{2cm} (27)

Applying the Laplace transformation, equation (1) changes to equation (28):

\[
c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial z^2} \right) = s \bar{u} - u(r, 0, z, 0)
\]  

(28)
Next step, finite Fourier Cosine transform is employed, so the equation turns into equation (29):

$$c^2 \left( \overline{u}_n + \frac{1}{r} \overline{u}_r + \frac{1}{r^2} \overline{u}_s - n^2 \pi^2 \overline{u}(r, 0, n, s) - 2\overline{u}_z(r, 0, 0, s) + (-1)^{n+1} \overline{u}_z(r, 0, h, s) \right) = s\overline{u}(r, 0, n, s)$$  \hspace{1cm} (29)

Boundary conditions at the bottom and the top of the cylinder are zero so the simplified equation is appeared as equation (30):

$$c^2 \left( \overline{u}_n + \frac{1}{r} \overline{u}_r + \frac{1}{r^2} \overline{u}_s - n^2 \pi^2 \overline{u}(r, 0, n, s) \right) = s\overline{u}(r, 0, n, s)$$  \hspace{1cm} (30)

Taking Periodic Fourier transformation and simplify the equation result:

$$c^2 \left( \overline{u}_n + \frac{1}{r} \overline{u}_r - \frac{m^2}{r^2} \overline{u} \right) = \left( s + (cn\pi)^2 \right) \overline{u}(r, m, n, s)$$  \hspace{1cm} (31)

In which:

$$\overline{u}(r, m, n, s) = \int_{-\pi}^{\pi} \overline{u}(r, 0, n, s)e^{-im\theta} d\theta$$

$$\overline{u}_n = \frac{\partial^2 u(r, 0, n, s)}{\partial r^2}$$

$$\overline{u}_r = -\frac{\partial u(r, 0, n, s)}{\partial r}$$

Considering 1 equal to zero in equation (11) and applying Hankel Transform the solution forms as equation (32):

$$\hat{u}(k, m, n, s) = 2c^2 \frac{\overline{u}(b, m, n, s)}{\pi} \left( \frac{1 + e^{-1}}{1 + n^2 \pi^2} \right)$$  \hspace{1cm} (32)

In which:

$$\overline{u}(b, m, n, s) = \frac{2T_b}{s} \left( -\frac{1 + e^{-1}}{1 + n^2 \pi^2} \right) \left( \frac{2\pi(1 - s)}{m} \right)$$

Equation (32) is considered in two parts: n = 0 and n > 0. For the first case taking inverse Hankel transform results equation (33):

$$\overline{u}(r, m, n, s) = \sum_{i=1}^{\infty} \frac{\pi^2}{2} J_m^2(ak_i) \left( \frac{2c^2}{\pi} \frac{1}{s + (cn\pi)^2 + (ck_i)^2} \right) \left( -\frac{1 + e^{-1}}{1 + n^2 \pi^2} \right) \left( \frac{2\pi(1 - s)}{m} \right) \left[ J_m(k, r)Y_m(k, b) - J_m(k, b)Y_m(k, r) \right]$$  \hspace{1cm} (33)

Using inverse Laplace, Periodic Fourier and Cosine Fourier transforms equation (33) changes to equation (34):

$$u(r, 0, z, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=-\infty}^{\infty} c^2 T_b \frac{k_i^2 J_m^2(ak_i)}{J_m^2(ak_i) - J_m^2(bk_i)} \left( \frac{1}{s + (cn\pi)^2 + (ck_i)^2} \right) \left( -\frac{1 + e^{-1}}{1 + n^2 \pi^2} \right) \left( \frac{2\pi(1 - s)}{m} \right) \left[ J_m(k, r)Y_m(k, b) - J_m(k, b)Y_m(k, r) \right] e^{im\theta} \cos(n\pi z) \left( 1 - e^{-\left( (cn\pi)^2 + (ck_i)^2 \right) t} \right)$$  \hspace{1cm} (34)

Moreover, the steady state answer is, equation (35):
\(u(r, \theta, z) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} c_i r^2 J_n^2(a_k) \frac{1}{J_m^2(a_k) - J_m^2(b_k)} \frac{1}{(\sin \pi)^2 + (ck_i)^2} \left( \frac{-1 + e(-1)^n}{1 + n^2 \pi^2} \right) \right) \frac{2\pi(-1)^m}{m} \) 

\[X[J_m(k, r)Y_m(k, b) - J_m(k, b)Y_m(k, r)]e^{im\theta} \cos(n\pi z)\]

Also the solution of the second part is, equation (36):

\[\frac{u(r, \theta, t)}{2} = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} T_i \frac{J_m^2(a_k)}{J_m^2(a_k) - J_m^2(b_k)} \frac{1}{(\sin \pi)^2 + (ck_i)^2} \left(1 - e^{-ck_i^2t} \right) \frac{2\pi(-1)^m}{m} \]

\[X[J_m(k, r)Y_m(k, b) - J_m(k, b)Y_m(k, r)]e^{im\theta} \cos(n\pi z)\]

And the steady state answer for this part is, equation (37):

\[\frac{u(r, \theta, 0)}{2} = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} T_i \frac{J_m^2(a_k)}{J_m^2(a_k) - J_m^2(b_k)} \frac{1}{(\sin \pi)^2 + (ck_i)^2} \left(e^{-1} \right) \frac{2\pi(-1)^m}{m} \]

\[X[J_m(k, r)Y_m(k, b) - J_m(k, b)Y_m(k, r)]e^{im\theta} \cos(n\pi z)\]

Finally, the analytical answer for steady-state situation is presented in equation (38):

\[\frac{u(r, \theta, 0)}{2} = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} T_i \frac{J_m^2(a_k)}{J_m^2(a_k) - J_m^2(b_k)} \frac{1}{(\sin \pi)^2 + (ck_i)^2} \left(e^{-1} \right) \frac{2\pi(-1)^m}{m} \]

\[X[J_m(k, r)Y_m(k, b) - J_m(k, b)Y_m(k, r)]e^{im\theta} \cos(n\pi z)\]

In order for simulating \(k_i\) should found by equation (39) [9]:

\[J_m(k, b)Y_m(k, a) - J_m(k, a)Y_m(k, b) = 0\]

In which \(a = 0.5, b = 1\) and \(-10 \leq m \leq 10\). In table (2) the roots of equation (39) are presented.

<table>
<thead>
<tr>
<th>m</th>
<th>-10.10</th>
<th>-9.9</th>
<th>-8.8</th>
<th>-7.7</th>
<th>-6.6</th>
<th>-5.5</th>
<th>-4.4</th>
<th>-3.3</th>
<th>-2.2</th>
<th>-1.1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_1)</td>
<td>14.5</td>
<td>13.40</td>
<td>12.31</td>
<td>11.23</td>
<td>10.18</td>
<td>9.19</td>
<td>8.26</td>
<td>7.45</td>
<td>6.81</td>
<td>6.93</td>
<td>6.24</td>
</tr>
<tr>
<td>(k_2)</td>
<td>18.82</td>
<td>17.80</td>
<td>16.84</td>
<td>15.93</td>
<td>15.10</td>
<td>14.37</td>
<td>13.74</td>
<td>13.23</td>
<td>12.85</td>
<td>12.62</td>
<td>12.54</td>
</tr>
<tr>
<td>(k_3)</td>
<td>23.57</td>
<td>22.74</td>
<td>21.97</td>
<td>21.27</td>
<td>20.65</td>
<td>20.11</td>
<td>19.66</td>
<td>19.30</td>
<td>18.94</td>
<td>18.88</td>
<td>18.83</td>
</tr>
<tr>
<td>(k_4)</td>
<td>28.84</td>
<td>27.55</td>
<td>27.00</td>
<td>26.51</td>
<td>26.09</td>
<td>25.74</td>
<td>25.47</td>
<td>25.28</td>
<td>25.16</td>
<td>25.12</td>
<td>25.12</td>
</tr>
<tr>
<td>(k_5)</td>
<td>34.45</td>
<td>33.89</td>
<td>33.38</td>
<td>32.93</td>
<td>32.53</td>
<td>32.19</td>
<td>31.91</td>
<td>31.69</td>
<td>31.53</td>
<td>31.43</td>
<td>31.40</td>
</tr>
<tr>
<td>(k_6)</td>
<td>40.26</td>
<td>39.78</td>
<td>39.35</td>
<td>38.97</td>
<td>38.63</td>
<td>38.34</td>
<td>38.11</td>
<td>37.93</td>
<td>37.79</td>
<td>37.71</td>
<td>37.69</td>
</tr>
<tr>
<td>(k_7)</td>
<td>46.2</td>
<td>45.78</td>
<td>45.40</td>
<td>45.07</td>
<td>44.78</td>
<td>44.54</td>
<td>44.33</td>
<td>44.18</td>
<td>44.06</td>
<td>44.00</td>
<td>43.97</td>
</tr>
<tr>
<td>(k_8)</td>
<td>52.4</td>
<td>51.84</td>
<td>51.51</td>
<td>51.22</td>
<td>50.97</td>
<td>50.75</td>
<td>50.57</td>
<td>50.43</td>
<td>50.33</td>
<td>50.28</td>
<td>50.26</td>
</tr>
<tr>
<td>(k_9)</td>
<td>58.28</td>
<td>57.96</td>
<td>57.66</td>
<td>57.40</td>
<td>57.17</td>
<td>56.98</td>
<td>56.82</td>
<td>56.70</td>
<td>56.61</td>
<td>56.56</td>
<td>56.54</td>
</tr>
<tr>
<td>(k_{10})</td>
<td>64.40</td>
<td>69.40</td>
<td>63.83</td>
<td>63.60</td>
<td>63.39</td>
<td>63.22</td>
<td>63.08</td>
<td>62.97</td>
<td>62.89</td>
<td>62.84</td>
<td>62.82</td>
</tr>
</tbody>
</table>
Considering $T_0 = 100^\circ C$ and $c = 1$, simulation results are obtained. In figure (7), the temperature field is shown and as it was expected; in this case, temperature is dependent on $\theta$ and with increasing that, temperature is enlarged. Temperature at $\theta = \pi, -\pi$ has different signs so there is a jump at this point. Roughness of the figure is caused by selecting small range for $m, n$. If these values are chosen in a wide range, the result will be smoother.

Figure (7): The temperature field at $z = 0.5$.

In order to verify the solution, finite element method is used and results the figure (8), the distribution of temperature at $z = 0.5$ and $\theta = \frac{\pi}{2}$.

Figure (8): The temperature field at $z = 0.5$ and $\theta = \frac{\pi}{2}$. 

767
And in figure (9) the variation of temperature at $r = 0.75$ and $\theta = \frac{\pi}{2}$ is presented. The verification is not as precisely as axis symmetric case and the reason is choosing the range of $-10 \leq m \leq 10$.

![Figure (9): The temperature field at $r = 0.75$ and $\theta = \frac{\pi}{2}$.](image)

Conclusion

In this paper, the heat conduction of a hollow cylinder was discussed and solved as an exact solution method. Using Generalized Finite Hankel Transform is systematic, simple and reliable as it was shown in verifications with Finite Element Method. Two situations, axis symmetric and general case, were explained and simulated. Finally, the results show that this method is suitable for solving PDE equations in cylindrical coordinates.

References


