A computer aided study of geodesics in the Schwarzschild Anti-de Sitter space-time

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ABSTRACT: The article illustrates the graphical study of geodesic motion on curved space-time using Maple platform.

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INTRODUCTION

Learning to solve the geodesic equations is an integral part of a first course on general relativity. However, most texts on general relativity do not contain a sufficient number of solvable examples that illustrate the behavior of geodesics. This lack is largely due to the fact that the geodesic equations are not easy to solve exactly. The use of Maple in general relativity and in the study of geodesic motion has a long history. See [2] for a list of articles using Maple and GrTensorII package. The article illustrates the graphical study of geodesic motion on Schwarzschild Anti-de Sitter space-time using the symbolic, and graphical computation facilities of Maple platform + GrTensorII.

Geodesics in the Schwarzschild Anti-de Sitter space-time

The metric for a static spherically symmetric spacetime with mass $M$ and a negative cosmological constant $\Lambda = -\frac{3}{l^2}$ is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin\theta^2 d\phi^2)$$

Where the lapse function, $f(r)$, is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2},$$

and the coordinates are defined such that $-\infty \leq t \leq +\infty$, $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

We can enter this metric directly in this form using ([1],[2]) makeg(), giving it the name `Ads':

```
> restart; grtw();
makeg(Asd):
```

To quit makeg, type 'exit' at any prompt.

Do you wish to enter a metric $[g(dn,dn)]$, line element $[ds]$, non-holonomic basis $[e(1)...e(n)]$, or NP tetrad $[l,n,m,mbar]$?

> 2;
> Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [t,r,theta,phi];
> Enter the line element using d[coord] to indicate differentials. (for example, r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
> [Type 'exit' to quit makeg]
> ds^2 = - (1-2*M/r+r^2/l^2)*d[t]^2 + d[r]^2/(1-2*M/r+r^2/l^2) + r^2* (d[theta]^2 + sin(theta)^2*d[phi]^2)
> makeg completed;

The Christoffel symbols in coordinates $(t, r, \theta, \phi)$ are

$$\gamma^r_{tt} = \frac{(r^2 - 2Mr + r^2)(Ml^2 + r^3)}{r^3 l^4}$$
Thus we obtained the next four geodesic equations:

\[ \begin{align*}
\gamma^t_{tr} &= \frac{\dot{M} t^2 + r^3}{r(t)^2 - 2M t^2 + r^3} \\
\gamma^\theta_{\theta \phi} &= \frac{1}{r^2}, \quad \gamma^\phi_{\theta \phi} = -\frac{r(t)^2 - 2M t^2 + r^3}{l(t)^2} \\
\gamma^\phi_{\phi \phi} &= \cot(\theta), \quad \gamma^\phi_{\theta \phi} = -\frac{(r(t)^2 - 2M t^2 + r^3)(\sin(\theta))^2}{l(t)^2} \\
\gamma^\phi_{\theta \theta} &= -\sin(\theta) \cos(\theta)
\end{align*} \]

Next we shall define the 4-velocity and 4-acceleration, namely

\[ \mathbf{v}^i = \frac{dx^i}{dt}; \quad \mathbf{a}^i = \frac{d^2x^i}{dt^2} \]

where \( \tau \) is the proper time measured in the proper reference frame of the parti- cle. Similar to [2] we shall use an intermediate coordinate set \( \mathbf{te}(\tau), \mathbf{er}(\tau), \mathbf{ph}(\tau), \mathbf{th}(\tau) \) instead of the coordinates \((t, r, \theta, \phi)\) fixes by GrTensolll, till we shall come back to Maple, where there is no possibility of confusion. We done this as a series of GrTensolll definitions, we use some ideas from [1], [2] in our calculation.

\[
\begin{align*}
> \text{grdef('v[tau]'):=[diff(te(tau),tau),diff(er(tau),tau),}
&\quad \text{diff(th(tau),tau),diff(ph(tau),tau)]}; \\
> \text{grcalc('v(up)'); grdisplay('v(up)');}
\end{align*}
\]

In this way we defined and computed a GrTensolll 4-vector, \( \mathbf{geo}(\mathbf{dn}) \) containing as components the four geodesic equations. The following commands calculate the next four geodesic equations:

\[
\begin{align*}
> \text{four:=grcomponent('geo(up)',[phi])}; \\
> \text{one:=grcomponent('geo(up)',[t])}; \\
> \text{two:=grcomponent('geo(up)',[r])}; \\
> \text{three:=grcomponent('geo(up)',[theta])}; \\
> \text{one:=subs(r=r(tau),one)}; \\
> \text{two:=subs(r=r(tau),two)}; \\
> \text{three:=subs(r=r(tau),three)}; \\
> \text{four:=subs(r=r(tau),four)};
\end{align*}
\]

Thus we obtained the next four geodesic equations:

\[
\begin{align*}
\frac{\frac{d^2}{dt^2} t(\tau)}{r(\tau)} \left( \frac{r(\tau)^2}{l(t)^2 - 2M l^2 + (r(\tau))^3} \right)^2 - 2 \frac{\frac{d^2}{dt^2} t(\tau)}{r(\tau)} M l^2 + \frac{\frac{d^2}{dt^2} t(\tau)}{r(\tau)} = 0 \\
2 \frac{\frac{d}{dt} t(\tau)}{r(\tau)} = 0 \\
- \frac{\frac{d^2}{dt^2} th(\tau)}{r(\tau)} - 2 \frac{\frac{d}{dt} er(\tau)}{r(\tau)} th(\tau) + \sin(\theta) \cos(\theta) \frac{\frac{d}{dt} ph(\tau)}{r(\tau)} = 0
\end{align*}
\]
\[
\left( \frac{d^2}{dt^2} ph(r) \right) r(\tau) \sin(\theta) + 2 \left( \frac{d}{dt} er(\tau) \right) \left( \frac{d}{dt} ph(r) \right) \sin(\theta) + 2 \cos(\theta) \left( \frac{d}{dt} th(r) \right) \left( \frac{d}{dt} ph(r) \right) r(\tau) \\
r(\tau) \sin(\theta) = 0
\]

The equation two is very tall so we replaced it by two.

We can solve the equation one using the next Maple commands:

```maple
> bau:=diff(((1-2*M/r(tau)+(r(tau))^2/l^2))*diff(t(tau),tau),tau);
> expand(simplify(one*(-1+2*M/r(tau)-(r(tau))^2/l^2)+bau));
> one:=expand(simplify(one*(-1+2*M/r(tau)-(r(tau))^2/l^2)));
> ecuone:=(1-2*M/r(tau)+(r(tau))^2/l^2)*diff(t(tau),tau)-C1;
> diffte:=solve(ecuone,diff(t(tau),tau));
> ecuone:=expand(simplify(subs(diff(t(tau),tau)=diffte,ecuone)));
> one:=expand(simplify(subs(diff(t(tau),tau)=diffte,one)));
> one:=expand(simplify(one));

obtaining for \[\frac{dt}{d\tau} = \frac{C_1 r(\tau) l^2}{r(\tau)^2 - 2 M l^2 + (r(\tau))^3}\]

where \[C_1\] is a constant. Now we will solve the four equation as:

```maple
> bau2:=diff(r(tau)^2*sin(theta(tau))^2*diff(phi(tau),tau),tau);
> expand(simplify(subs(sin(theta(tau))=sin(theta),cos(theta(tau))=cos(theta),bau2/r(tau)^2*sin(theta(tau))^2-four));
> ecufour:=r(tau)^2*sin(theta)^2*diff(phi(tau),tau)-C2;
> diffph:=solve(ecufour,diff(phi(tau),tau));
> ecufour:=expand(simplify(subs(diff(phi(tau),tau)=diffph,ecufour)));
> four:=expand(simplify(subs(diff(phi(tau),tau)=subs(theta=theta(tau),diffph),subs(cos(theta)=cos(theta(tau)),sin(theta)=sin(theta(tau)),four))));
> two:=subs(diff(t(tau),tau)=diffte,diff(phi(tau),tau)=diffph,three);
> three:=subs(diff(phi(tau),tau)=diffph,three);

Obtaining for \[\frac{d\theta}{d\tau} = \frac{C_2}{(r(\tau))^2 (\sin(\theta))^2}\]

where \[C_2\] is a constant. To solve the last two remaining equation (two and thus we have: three) is necessary to fix \[\theta = \frac{\pi}{2}\].

```maple
> bau3:=diff(r(tau)^2*diff(theta(tau),tau),tau)-r(tau)^2*sin(theta)*cos(theta)*diffph^2;
> expand(simplify(bau3/r(tau)^2-three,trigsin));
> two;
> three;
> theta(tau):=Pi/2;
> two:=eval(subs(theta=Pi/2,eval(two)));
> three:=eval(subs(theta=Pi/2,eval(three)));
```

We observe that the second equation (three) is now canceled by the fixing of \[\theta\] coordinate. Finally we have the equation two. Our purpose is from now one to split the above equation in two different ones, expressing the derivatives for \[r\] and \[t\] in terms of the angular coordinate \[\phi\].
We obtained the next two differential equations:

\[ \text{two 1} = 0 \] (2)

\[ \text{two 2} = 0 \] (3)

We used the Maple commands sequence below for numerical integration of the system of equations (2,3) we used \(m = 1, C_2 = \pi, C_1 = 1.\)

```maple
> with(DEtools):
> two1graph:=subs(C2=Pi,C1=1,M=1,l=10,two1);
> ini1:=r(0)=3,D(r)(0)=1;
> two2graph:=subs(M=1,C2=Pi,C1=1,two2);
> ini2:=t(0)=0;
> DEplot(two1graph,r(phi),phi=0..Pi,[[ini1]],method=classical,
axes=BOXED,thickness=5,stepsize=0.001);
> DEplot3d([two1graph,two2graph],[r(phi),t(phi)],-Pi/4..Pi,[[ini1,ini2]],
method=classical,stepsize=0.001,axes=BOXED,thickness=5);
```

**REFERENCES**


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