Longitudinal Velocity Distribution in Compound Open Channels: Comparison of Different Mathematical Models

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ABSTRACT: Measuring the mean velocity profile through the channel cross section is one of basic issues in hydraulic modeling. In wide open channels, the log law and the log-wake are suitable for describing the velocity profile, but in case of narrow open channel flow, such models deviate from experimental data near the free surface. In such open channels, the three dimensional nature of flow, and transport momentum from the side walls to the central zone, causes the maximum velocity to occur below the water surface producing the velocity-dip-phenomenon. Therefore, developing new relations which is able to show this phenomenon seems to be necessary. Numerous models for determination of velocity profile in narrow open channels have been proposed by many researchers. This study provides a comparison among various existing methods for computing the longitudinal velocity distribution in narrow channels. Four models of velocity distribution were selected such as Yang’s, Bonakdari’s, Absi’s, and Kundu and Ghoshal’s model. To evaluate the accuracy of the models in calculating the velocity profiles, the results of the models were compared to the experimental data obtained from an actual measurement site in narrow sewer. Sewers are designed on the basis of open-channel-flow principles. The results show that all models represent good agreement with the experimental data, however, Absi’s model and Kundu and Ghoshal’s model are more successful in predicting the position of maximum velocity, and provide the least errors in most cases. Also it is concluded that for lower aspect ratios (which makes the condition of narrow channels be satisfied better), the results of the models tend to coincide better with the measured data.

Key words: sewer, compound open channel, velocity distribution, dip phenomenon

INTRODUCTION

In open channel flows, the velocity profile is well described by the conventional log-law in the inner region (\( y < 0.2D \)); but in the outer region of the flow (\( y > 0.2D \)), the log-law deviates from experimental data. The log-wake law (which is of the most accepted laws for describing the velocity profile in the outer region) modifies the log-law by adding a wake function (Coles, 1956). Previous studies used to concentrate on two-dimensional flows where secondary currents can be ignored and the maximum velocity occurs at the free surface. For such flows, the log law and the log-wake law can describe the velocity profiles well (Steffler et al., 1985); (Nezu and Rodi, 1986); (Kirkgoz, 1989); (Cardoso et al., 1989); (Kironoto and Graf, 1994); (Muste and Patel, 1997). But in case of narrow open channels [where the aspect ratio of the channel width to flow depth is less than five (\( 5 < D_B \))], the existence of secondary currents generated due to the 3-D nature of the flow, the maximum velocity occurs below the water surface, and therefore, there is a negative gradient of velocity distribution near the free surface, which is called the velocity-dip-phenomenon. In such cases, the primary models deviate from the experimental results near the free surface and fail to predict Dip phenomenon, as they impose a velocity which increases monotonically with increase of distance from the bed(Absi, 2011), (Kundu & Ghoshal, 2012).

Modeling the dip phenomenon is important for numerous hydraulic applications such as developing stage-discharge relationships, analysis of contaminant transport, and to define the relationship between surface and mean flow velocities (Lee and Julien 2006). The velocity dip phenomenon was first reported more than a century
ago. Since that time, numerous investigations have been conducted by many researchers (Sarma et al., 1983, 2000), (Yang et al., 2004), (Guo & Julien, 2005), (Chiu, 2006), (Bonakdari, 2006), (Absi, 2011), (Moazzamnia & Bonakdari, 2012), (Kundu & Ghoshal, 2012), (Bonakdari & Ahadi, 2013), (Lassabatare et al., 2013), (Cui & Singh, 2013), in order to propose new models to be able to not only describe the dip phenomenon and negative gradient of velocity near the free surface, but also to predict the position of the maximum velocity accurately and fit the experimental data throughout the whole flow depth.

The objective of this paper is to examine the ability of several different models in describing the velocity distribution profile in narrow open channels as well as predicting the position of maximum velocity below the water surface.

Data obtained from an actual open-channel sewer are used to evaluate the accuracy of the models in predicting the longitudinal velocity profiles.

Introduction to the models

In this study, four models for calculating velocity distribution in narrow open channels are discussed and compared both with each other and the measured data from an actual sewer channel. The reason why these five models have been chosen to be studied in this work is that all of them are based on different relations available for eddy viscosity, and are capable of describing dip phenomenon and velocity negative gradient near the free surface. Also they are simple and easy to use for computing velocity distribution in open channels.

Yang et al. (2004): Dip-Modified-Log law

For steady uniform open-channel flows, using the continuity equation, the RANS momentum reads in the streamwise direction \( x \) (Fig. 1) as:

\[
\frac{\partial U V}{\partial y} + \frac{\partial U W}{\partial z} = \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial y} - \frac{\partial U}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial U}{\partial z} - \frac{\partial U}{\partial y} \right) + g \sin \theta
\]

(1)

where \( x, y, \) and \( z \) are streamwise, vertical, and lateral directions respectively and \( U, V, \) and \( W \) the corresponding mean velocities with \( u, v, \) and \( w \) as turbulent fluctuations, \( \nu \) is the fluid kinematic viscosity, \( g \) the gravitational acceleration, and \( \theta \) is the angle of the channel bed to the horizontal (Fig. 1). By considering \( S = \sin \theta \) as channel bed slope, Eq. (1) may be written as:

\[
\frac{\partial}{\partial y} \left( U V - \frac{\nu \partial U}{\partial y} \right) + \frac{\partial}{\partial z} \left( U W - \frac{\nu \partial U}{\partial z} \right) = g S
\]

(2)

In the central area of the channel (see Fig. 1), it is assumed that the vertical gradients \( \left( \frac{\partial}{\partial y} \right) \) are dominating, which allows to ignore the horizontal gradients \( \left( \frac{\partial}{\partial z} \right) \) (Yang et al. 2004).

Since for large values of \( y \) the viscous part \( \left( \frac{\nu \partial U}{\partial y} \right) \) of the shear stress \( \tau / \rho = \left( \frac{\nu \partial U}{\partial y} \right) - \overline{uv} \) (where \( \rho \) is fluid density), is small versus the turbulent part \( -\overline{uv} \) (Absi, 2008), Eq. (2) becomes:

\[
\frac{\partial U V}{\partial y} + \frac{\partial U W}{\partial y} = g S
\]

(3)

Integration of Eq. (3) gives:
\[-\frac{\overline{uv}}{u_*} = \left(1 - \frac{y}{D}\right) - \alpha_i \frac{y}{D} + \frac{UV}{u_*} \quad \text{(4)}\]

where \( u_* \) is friction velocity and \( \alpha_i = \left[ gSD/u_*^2 \right] - 1 \). By assuming (Yang et al. 2004):

\[ \frac{UV}{u_*} \approx -\alpha_2 \frac{y}{D} \quad \text{(5)}\]

where \( \alpha_2 \) is a positive coefficient, Eq. (4) becomes:

\[ -\frac{\overline{uv}}{u_*} = \left(1 - \frac{y}{D}\right) - \alpha \frac{y}{D} \quad \text{(6)}\]

Where \( \alpha = \alpha_i + \alpha_2 \). With the Boussinesq assumption (Boussinesq, 1887):

\[ -\frac{\overline{uv}}{\nu} = \frac{dU}{dy} \quad \text{(7)}\]

Equation (6) gives:

\[ \frac{dU}{dy} = u_* \left(1 - \frac{y}{D}\right) - \alpha \frac{y}{D} \quad \text{(8)}\]

Yang et al. (2004) used a widely used parabolic eddy viscosity:

\[ \nu_r = ku_0 \left(1 - \frac{y}{D}\right) \quad \text{(9)}\]

where \( \kappa \approx 0.41 \) is the von Karman constant, thus Eq. (8) can be written as:

\[ \frac{dU}{dy} = \frac{u_*}{\kappa} \left(1 - \frac{y}{D}\right) \quad \text{(10)}\]

Integration of Eq. (10) gives:

\[ \frac{U}{u_*} = \frac{1}{k} \ln\left(\frac{y}{y_0}\right) + \alpha \ln\left(1 - \frac{y}{D}\right) + \ln\left(1 - \frac{y_0}{D}\right) \quad \text{(11)}\]

Where \( y_0 \) is the distance at which the velocity is assumed to be equal to zero. Since \( y_0/D \ll 1 \) and with \( U_a = \frac{U}{u_*} \) and \( \xi = \frac{y}{D} \) and \( \xi_0 = \frac{y_0}{D} \), Eq. (11) simplifies to (Yang et al., 2004):

\[ U_a = \frac{1}{k} \ln\left(\frac{\xi}{\xi_0}\right) + \alpha \ln(1 - \xi) \quad \text{(12)}\]

Eq. (12) predicts the velocity-dip-phenomenon by the term \( \ln(1 - \xi) \) and \( \alpha \) as dip-correction parameter (Yang et al. 2004). Eq. (12) contains only \( \alpha \) and reverts to the classical log law for \( \alpha = 0 \). Yang et al. (2004) proposed the empirical relation:

\[ \alpha(z) = 1.3 \exp(-z/D) \quad \text{(13)}\]

where \( z \) is the lateral distance from the side wall.

**Bonakdari’s Model (2008)**

Bonakdari et al. (2006, 2007, 2008) developed a new model for computing the vertical velocity profile in the central region of steady, fully developed, turbulent open-channels with turbulent flow. The coordinates used in this model is shown in fig. 2.
For steady uniform and fully developed open channel flows, using the continuity equation, the momentum equation in longitudinal direction \((x)\) reads as:

$$\frac{\partial (U V)}{\partial y} + \frac{\partial (U W)}{\partial z} = g \sin \theta + \frac{\partial}{\partial y} (-uv) + \frac{\partial}{\partial z} (-uw) + v \frac{\partial^2 U}{\partial y^2} + v \frac{\partial^2 U}{\partial z^2}$$

(14)

In which \(g\) is the gravity acceleration and \(\sin \theta\) is the energy slope. Using some simplifications, and considering the non-slip boundary condition, \((u=0)\) at \(\xi = \xi_0\), where \(\xi_0 \ll 1\), finally the distribution of the mean velocity in the central channel, and out of the inner region, \((\xi > 0.2)\) can be expressed as:

$$\frac{U}{u_*} = -\frac{\beta}{k} \frac{1}{(0.5\xi^2 + \xi + C(A))} (0.25\xi^2 + \xi + C(A) \ln \frac{\xi}{\xi_0})$$

(15)

where \(U\) is the longitudinal velocity, \(u_*\) shear velocity at the bottom, \(\beta = \frac{g D \sin \theta}{u_*^2}\), \(\xi = \frac{y}{D}\), and \(C(A)\) is a parameter.

This formulation is able to predict the velocity distribution in the outer region of the turbulent boundary layer in the central zone of an open channel. The proposed law is a modification of the well-known logarithmic law, which involves an additional parameter, depending on the position of the maximum velocity and the roughness height (Bonakdari, 2007).

The advantage of this model is that it is suitable for narrow and wide-open channels, and its parameters can be calculated based on channel geometry and hydraulic characteristics of the flow.

**Absi (2011): full Dip-Modified-Log-Wake law**

Absi (2011) inserted a more appropriate approximation for eddy viscosity in accordance with the log-wake law given by Nezu and Rodi (1986) instead of the parabolic profile for eddy viscosity (Eq. 9); and obtained the ordinary differential equation for velocity distribution as:

$$\frac{dU}{dy} = \frac{u_*}{k D} \left( 1 - \alpha \frac{y}{D} \right) \left[ \frac{D}{y} + \pi \ln(\frac{D}{y}) \right]$$

(16)

Finally by integrating the dimensionless form of Eq. (16) for \(\xi_0 << 1\), the full dip-modified-log-wake law (IDMLW-law) was developed (Absi, 2011):

$$U_a = \frac{1}{k} \ln(\frac{\xi}{\xi_0}) + \frac{2 \pi}{k} \sin^2(\frac{\pi \xi}{2}) + \frac{\alpha}{k} \ln(1-\xi) - \frac{\alpha \pi}{k} \int_{\xi_0}^{\xi} \frac{\xi}{1-\xi} \sin(\pi \xi) d\xi$$

(17)

The last term of Eq. (17) needs to be integrated using the trapezoidal or Simpson rules. Parameter \(\alpha\) can be calculated either from suggested relation by Yang (2004) as mentioned before, or by the following Equation proposed by Absi, (2011):

$$\alpha = \frac{1}{\xi_{dip}} - 1$$

(18)

in which \(\xi_{dip}\) is the dimensionless distance from the bed corresponding to the maximum velocity.
Kundu & Ghoshal (2012): total Dip-Modified-Log-Wake law
With some assumptions and simplifications, Kundu and Ghoshal (2012) expressed the shear stress distribution as follows:
\[
\frac{\tau \alpha}{\tau_0} = \left(1 - \frac{y}{D}\right) - \alpha \frac{y}{D}
\]
(19)
in which \(\alpha\) is a positive constant which is the dip-correction parameter (Yang et al., 2004). From Eq. (19) and using Boussinesq’s model (Boussinesq et al., 1877) which relates the Reynolds shear stress with the strain rate by introducing the eddy viscosity \(\nu_t\), one can get:
\[
\frac{du}{dy} = \frac{u^2}{\nu_t} \left[\left(1 - \frac{y}{D}\right) - \alpha \frac{y}{D}\right]
\]
(20)
Unlike the previous models which mostly use either the parabolic profile or the equation proposed by Nezu & Rodi (1986), Kundu and Ghoshal (2012) offered a new relation for eddy viscosity as:
\[
\frac{\nu_t}{u_D} = k(1 - \xi)[1 + 12 \Pi \xi(1 - \xi)]^{-1}
\]
(21)
Inserting Eq. (21) into Eq. (20), and integrating for \(\xi_0 << 1\), they finally obtained the following relation for velocity distribution in uniform open channel flows:
\[
\frac{U}{u_*} = \frac{1}{k} \ln\frac{\xi}{\xi_0} + \frac{\alpha}{k} \ln(1 - \xi) + \frac{2\Pi}{k} (3\xi^2 - 2\xi^3) - \frac{4\alpha\Pi}{k} \xi^3
\]
(22)
Eq. (22) is referred to as the total-dip-modified-log-wake law (TDMLWL). The advantage of this model is that it is fully analytical, and for a condition where \(\alpha\) and \(\Pi\) are equal to zero, it gives the classical log law. Parameter \(\alpha\) can be calculated by Equation (18). Parameter \(\alpha\) can be obtained from Eq. (18).

Field data
The experimental data collected from an actual site have been used in this study in order to evaluate the accuracy of the models in calculating velocity profiles in narrow open channels.

The chosen experimental site is located in a region called Jarden des Plantes, a few kilometers upstream from the treatment plant on the main sewer line of the City of Nantes (northwestern France). The cross section corresponds to a narrow section made of an egg-shaped channel with a bank (Bonakdari, 2006) which makes a compound cross section. The cross section of the channel as well as the verticals on which velocity was measured are shown in Fig. 3(a). An experimental device, Cerberes, developed by Larrarte (Larrarte, 2006) allows operations from the ground level in order to measure the velocity field in the entire cross-section (Fig. 3(b)).

Figure 3. (a) Scheme of cross section of Jardin des Plantes channel, and (b) “Cerberes”: the 2D remote controlled device during measurement (Bonakdari et al., 2008)
RESULTS AND DISCUSSION

The velocity profiles calculated by the four models have been plotted for three water levels and six different lateral distances from the sidewall in each case, along with the experimental data obtained from the sewer channel (Figs. 4-6). Tables 1, 2 and 3 represent the RMSE calculated for each model in Jardin des Plantes sewer channel for different water levels and different B/D ratio (where B is the channel width and D is the flow depth).

Figure 4. Comparison of velocity distribution profiles calculated from different models (flow depth=0.54m)

Fig. 4 shows the velocity profiles by the models in the cross section of Jardin des Plantes channel sewer for the dry weather condition (i.e. low water level). From Fig. 4, it is clear that all four models proposed by Yang et al., Bonakdari et al., Absi, and Kundu and Ghoshal agree with the experimental data acceptably. Yang’s model is able to show velocity negative gradient near the free surface, however there are some deviations from the experimental data in the upper zone of the flow for some lateral distances from sidewall, and it seems to underestimate the velocity in the region near free surface, which makes the maximum velocity position to be estimated a little lower than what it really is based on the measured data. Bonakdari’s model shows a good fitness with the experimental results in the outer region, but in the inner region it deviates from the experimental results in most cases. The justification can be the fact that Bonakdari’s model was developed only for the central region of the flow, and it is not expected to be able to show the exact velocity distribution in the inner region. Absi’s model (2011) acts well throughout the whole depth and in most cases, fits the measured data acceptably. Also the profiles obtained by Absi’s model show the velocity-dip phenomenon and velocity negative gradient near the free surface better than Bonakdari’s model, especially in the central zone of the cross section. However, near the sidewalls, Bonakdari’s model acts best of all in the outer region of the flow. Kundu and Ghoshal’s model appears to offer a good description of velocity distribution through the depth of the channel and is able to show the negative slope of the velocity profile near the free surface. But this model is not as successful as Absi’s in predicting the position
where maximum velocity occurs. As can be seen in table 1, the most and least errors belong to Yang’s and Kundu and Ghoshal’s models respectively.

### Table 1. The RMSE calculated for the profiles plotted in Fig. 4

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<tr>
<td>2.78</td>
<td>0.54</td>
<td>0.1705</td>
<td>0.0559</td>
<td>0.0793</td>
<td>0.0493</td>
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Fig. 5 depicts the velocity profiles plotted by the models for D=0.6m. All models represent a good description of velocity profiles, but among them, Absi’s model and Kundu and Ghoshal’s model agree the best with the measured data throughout the whole depth. Also, these two models as well as Yang’s model make a good performance in depicting the velocity negative gradient near the free surface and the position of dip phenomenon, though in some cases (especially for Z=0.3m and Z=0.4m) it seems that Yang’s model underestimate the velocity near the flow surface. At this water level Bonakdari’s model show some deviations from the experimental results in some lateral distances from the sidewalls. As can be seen in table 2, Absi’s and Kundu and Ghoshal’s models represent the least errors compared to the measured data in the channel sewer.

### Table 2. The RMSE calculated for the profiles plotted in Fig. 5

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<tr>
<td>2.5</td>
<td>0.6</td>
<td>0.1549</td>
<td>0.1805</td>
<td>0.0739</td>
<td>0.0415</td>
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Fig. 6. is representative of the longitudinal velocity distributions obtained from the models at the maximum water level (D=0.77m). In this situation, the condition of the channel approaches more to narrow channel condition; thus the results of the models are supposed to be more relevant in comparison to low water level. As the water level increases, the position of maximum velocity lies deeper in the flow and the dip phenomenon is expected to be predicted more obviously by the models. It is clear from Fig. 6 that all models agree well with the experimental results and present a good description of velocity profile through the whole cross section and different lateral distances, as well as both inner and outer regions of the flow, and are capable of showing the velocity negative gradient near the free surface. However in some cases, Yang’s model and Bonakdari’s model deviate from the measured data in the inner region. This deviation is expected by Bonakdari’s model, since this model is proposed only for the central zone of the cross section of the flow. Absi’s model and Kundu and Ghoshal’s model perform the best in describing the velocity distribution throughout the whole depth. Among all models, Yang’s model predicts the position of maximum velocity below the water surface the best for the maximum water level in Jardin des Plantes open-channel sewer. However, the results by Yang’s model deviate from measured data near the free surface which makes larger errors. Table 3 presents RMSE for the velocity profiles plotted in Fig. 6.

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<tr>
<td>1.95</td>
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<td>0.2707</td>
<td>0.0999</td>
<td>0.089827</td>
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### CONCLUSION

Velocity distribution equations for open channel flows by four models which are based on different eddy viscosity distributions were discussed and compared in this study. To evaluate the accuracy of the models in predicting longitudinal velocity profiles in the cross section of narrow channels, experimental data collected from a natural sewer channel were used. Sewers are designed on the basis of open-channel flow principles, and therefore such data can be used for the purpose of this study. The results showed that all models made a good performance
in describing the dip phenomenon and velocity negative gradient near the free surface, and agree with the experimental data acceptably. Nevertheless, it can be said that Absi's model predicted the position of maximum velocity more exactly. Yang's model offered the largest errors for any water level in the sewer-channel, since it underestimates the velocity near the free surface and seems to exaggerate the negative slope of velocity profile near the surface of the flow. Kundu & Ghoshal's model presented the least errors in describing the velocity distribution profiles and agreed well with the experimental results in the sewer channel, though it seems to overestimate the velocity near the free surface compared to Absi’s model. However this problem can be solved by adjusting the parameters in the model. Parameter $\Pi$ plays an important role in describing the curve of velocity profile in narrow channels. In this study, Coles' wake strength parameter ($\Pi$) assumed to be positive, while it is known that Coles’ parameter may be considered negative. Likewise, the dip correction parameter ($\alpha$) (which is calculated based on different relations for the models investigated in this study) has a great influence on describing the velocity-dip phenomenon and negative gradient of velocity near the free surface. Also the results indicated that the nearer the condition of sewer-channel became to the condition of narrow open channels, the better were the predictions by the models. Considering all these facts, it can be concluded that all these models are capable of calculating the velocity profiles in the cross section of compound open-channels, however Absi's model and Kundu and Ghoshal's model performed better in describing the velocity profile throughout the whole depth of the flow. Absi’s model presented the best results in predicting the exact position of maximum velocity, and Kundu and Ghoshal’s model provided the least errors compared to the rest of models discussed in this study.

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