Two step Runge-Kutta-like method for Numerical Solutions of Fuzzy Differential Equations

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ABSTRACT: In this paper, we apply a numerical algorithm for solving the fuzzy first order initial value problem, based on two-step Extended Runge-Kutta-like formulae of order 3 (TSERKO3). We use Seikkala’s derivative. The elementary properties of this method are given. We use the extended Runge-Kutta-like formulae in order to enhance the order of accuracy of the solutions.

Key words: Fuzzy numbers; Fuzzy differential equations; Extended Runge-Kutta; Euler method

INTRODUCTION

The theoretical framework of fuzzy initial value problems (FIVPs) has been an active research field over the last few years. The concept of fuzzy derivative was first introduced by Chang and Zadeh (1972). It was followed up by Dubois and Prade (1982), who defined and used the extension principle. A comprehensive approach to FIVPs has been the work of Seikkala (1987), and Kaleva (1987,1990), especially in its generalized form given by Buckley and Feuring (2000). Their work is important as it overcomes the existence of multiple definitions of the derivative of fuzzy functions, see (Dubois et al. 1982), (Goetschel et al. 1986), (Ma et al. 1999), (Puri et al. 1983) and (Seikkala 1987). Also, Buckley et al. (2000) compares various solutions to the fuzzy initial value problem that one may obtain using different derivatives. The results of (Seikkala 1987) on a certain category of fuzzy differential equations (FDEs) have inspired several authors who have applied numerical methods for the solution of these equations. The most important contribution on these numerical methods is the Euler method provided by Ma et. al. (1999). Abbasbandy and Allahviranloo (2004) developed four-stage order Runge-Kutta methods for a Cauchy problem with a fuzzy initial value. Also, Palligkinis et al. (2009), the authors applied Runge–Kutta methods for a more general category of problems, and they proved convergence for s-stage Runge-Kutta methods. Pederson and Sambandham (2007, 2008) studied numerical methods for the hybrid fuzzy differential equations by Euler and Runge-Kutta methods. The authors Efatti and Pakdaman (2010) solved fuzzy differential equations by artificial neural network approach.

The rest of the paper is organized as follows. In Section 2, we will give some necessary notations and definitions of fuzzy set theory, fuzzy differential equations, and a family of extended Runge-Kutta-like formulae. In Section 3, a fuzzy Cauchy problem is defined. The numerically solved FIVPs by the third-order fuzzy Runge-Kutta-like formulae, two-step Euler and Runge-Kutta methods are introduced in Section 4 and 5, respectively. Numerical experiments are provided in Section 6 and the conclusion is in Section 7.

Preliminaries

The basic definitions of fuzzy numbers are given by Cong-Xin et al. (1991) and Goetschel et al. (1986). In this section, we review some of them.

2.1. Definitions and notations

Definition 2.1. A fuzzy number is a fuzzy set \( u : \mathbb{R} \rightarrow [0, 1] \) which satisfies

1. \( u \) is upper semicontinuous,
2. \( u(x) = 0 \) outside some interval \([c, d]\),
3. there are real numbers \( a, b : c \leq a \leq b \leq d \) for which
   - \( u(x) \) is monotonic increasing on \([c, a]\),
   - \( u(x) \) is monotonic decreasing on \([b, d]\), and
3.3 \( u(x) = 1, \ a \leq x \leq b \)

An equivalent parametric definition is also given by Cong-Xin et al. (1991) and Goetschel et al. (1986) as follows:

**Definition 2.2.** A fuzzy number \( u \) in parametric form is a pair \( u = (u(r), \bar{u}(r)) \), \( r \in [0, 1] \); which satisfies the following requirements:

1. \( u(r) \) is a bounded left continuous monotonic increasing function over \([0, 1]\),
2. \( \bar{u}(r) \) is a bounded left continuous monotonic decreasing function over \([0, 1]\), and
3. \( u(r) \leq \bar{u}(r), \ 0 \leq r \leq 1 \).

A crisp number \( \alpha \) is simply represented by \( u(r) = \bar{u}(r) = \alpha, \ 0 \leq r \leq 1 \).

A triangular fuzzy number, \( \mathbf{v} \), is defined by three numbers \( a_1 < a_2 < a_3 \) where the graph of \( v(x) \), the membership function of the fuzzy number \( \mathbf{v} \), is a triangle with base on the interval \([a_1, a_3]\) and vertex at \( x = a_2 \). We specify \( v \) as \((a_1/a_2/a_3)\). We will write:

1. \( v > 0 \) if \( a_1 > 0 \);
2. \( v \geq 0 \) if \( a_1 \geq 0 \);
3. \( v < 0 \) if \( a_3 < 0 \); and
4. \( v \leq 0 \) if \( a_3 \leq 0 \).

Let \( E \) be a set of all upper semi-continuous normal convex fuzzy numbers with bounded \( r \)-level intervals. It means that if \( v \in E \), then the \( r \)-level set

\[ [v]_r = \{ s|v(s)| \geq r \} \]

is a closed bounded interval which is denoted by \([v]_r = [v_1(r), v_2(r)]\).

**Lemma 2.1** Let \( v, w \in E \) and \( s \) be scalar. Then, for \( r \in (0, 1) \)

\[
[v + w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)] \\
[v - w]_r = [v_1(r) - w_1(r), v_2(r) - w_2(r)] \\
[v \cdot w]_r = [\min\{v_1(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\}, \max\{v_1(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\}], \\
[sv]_r = s[v]_r.
\]

The addition and scalar multiplication of fuzzy numbers are defined by the extension principle Dubois (1982). For arbitrary \( u = (u, \bar{u}), v = (v, \bar{v}) \) and \( k \in \mathbb{R}^1 \)

\[(u + v)(s) = \sup_{x = s \pm t} \min\{ u(s), v(t) \}, \quad (ku)(s) = u \left( \frac{s}{k} \right), \quad k \neq 0.\]

The collection of all fuzzy numbers with addition and multiplication is denoted by \( E^1 \) and is a convex cone. The fuzzy numbers of Def. (2.2) as shown in Cong-Xin et al. (1991) can be embedded into the Banach space \( B = C[0, 1] \times C[0, 1] \), where the norm is defined as

\[ ||(u, v)|| = \sup\{ \max_{r \in [0, 1]} |u(r)|, \max_{r \in [0, 1]} |v(r)| \}. \]  

(2.1)

**Definition 2.3.** For arbitrary fuzzy numbers \( u = (u(r), \bar{u}(r)) \) and \( v = (v(r), \bar{v}(r)) \), the quantity

\[ D(u, v) = \sup_{r \in [0, 1]} \max\{|u(r) - v(r)|, |\bar{u}(r) - \bar{v}(r)|\}, \]  

(2.2)

is the distance between \( u \) and \( v \).

The function \( D(u, v) \) is a metric on \( E^1 \). This metric function is equivalent to the one used by Puri and Ralescu (1983) and Kaleva (1987). It is shown Puri (1986) that \((E^1, D)\) is complete metric space.

**Definition 2.4.** A function \( f: \mathbb{R} \rightarrow E^1 \) is called a fuzzy function. If for arbitrary fixed \( t_0 \in \mathbb{R} \) and \( \varepsilon > 0, \delta > 0 \) such that

\[ |t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon \]

exists, \( f \) is said to be continuous.

The concept of fuzzy differentiation was introduced by Dubois and Prade (1982). Puri and Ralescu (1983) proposed two approaches for finding the fuzzy derivative. The first, based on the H-difference notation, regards \( E^1 \) as a universe. The second approach was also suggested by Goetschel and Voxman (1986).

Suppose that \( y: I \rightarrow E^1 \) is a fuzzy function. The parametric form of \( y(t) \) is represented by

\[ [y(t)]_r = [y_1(t, r), y_2(t, r)], \quad t \in I, \ r \in (0, 1), \]  

(2.4)

where \( I \) is a real interval. The Seikkala (1987) derivative \( y'(t) \) of fuzzy function \( y(t) \) is defined by
provided that this equation defines a fuzzy number.

Throughout the paper, we have used the Seikkala derivative.

Extended Runge-Kutta-like formulae

Consider the autonomous initial value problem
\[
\frac{dy}{dt} = f(y), \quad a \leq t \leq b, \quad y(a) = \alpha. \tag{2.6}
\]
We assume that \( f(y) \) has derivatives to the desired order in a domain \( D \) in \( \mathbb{R}^n \) and we assume that \( \| f(y_1) - f(y_2) \|_2 \leq L \| y_1 - y_2 \|_2 \)
holds for all \( y_1, y_2 \in D \), where \( L \) is the Lipschits constant. Many efforts have been made to improve the order of Runge-Kutta methods by means of increasing the numerical terms in Taylor series expansion. This increases the number of function evaluations accordingly (Chakrivate et al. 1983), (Enright, 1974), (Gear, 1971), (Hairer, 1991) and (Rosenbrock, 1963). Recently, Goeken et al. (2000) and Wu and Xia (2006) proposed a class of Runge-Kutta methods using higher derivatives and presented new third, fourth and fifth order numerical methods. Specifically, \( f' \) is embedded in \( f \), i.e., \( f' \) is approximated by a difference quotient of past and current evaluations of \( f, f'(y) \approx (f(y_n) - f(y_{n-1}))/h. \) This motivates a family of extended Runge-Kutta-like formulae of the form
\[
y_{n+1} = y_n + h \sum_{j=1}^{m} b_j k_j^{(1)} + h^2 \sum_{j=1}^{m} c_j k_j^{(2)}, \tag{2.7}
\]
where
\[
k_j^{(1)} = f \left( y_n + h \sum_{s=1}^{j-1} a_{js} k_s^{(1)} \right), \quad j = 1, 2, ..., m. \tag{2.8}
\]
\[
k_j^{(2)} = f' \left( y_n + h \sum_{s=1}^{j-1} b_{js} k_s^{(1)} \right), \quad j = 1, 2, ..., m. \tag{2.9}
\]
Obviously, with \( c_j = 0 \) (\( j = 1, 2, ..., m \)) in (2.7), the methods reduce to classical Runge-Kutta methods
\[
y_{n+1} = y_n + h \sum_{j=1}^{m} b_j k_j, \tag{2.10}
\]
where
\[
k_j = f \left( y_n + h \sum_{s=1}^{j-1} a_{js} k_s \right), \quad j = 1, 2, ..., m. \tag{2.11}
\]
We also note that, if \( a_{js} = b_{js}, j = 1, 2, ..., m, s = 1, 2, ..., j - 1 \) in (2.7)-(2.9), then we have
\[
y_{n+1} = y_n + h \sum_{j=1}^{m} b_j k_j^{(1)} + h^2 \sum_{j=1}^{m} c_j k_j^{(2)}, \tag{2.12}
\]
where
\[
k_j^{(1)} = f \left( y_n + h \sum_{s=1}^{j-1} a_{js} k_s^{(1)} \right), \tag{2.13}
\]
\[
k_j^{(2)} = f' \left( y_n + h \sum_{s=1}^{j-1} b_{js} k_s^{(1)} \right), \quad j = 1, 2, ..., m. \tag{2.14}
\]

Third-order formulae

Extended Runge-Kutta-like methods (2.7)-(2.9) with \( m = 2 \) are of the following form:
\[
y_{n+1} = y_n + h \left( b_1 k_1^{(1)} + b_2 k_2^{(1)} \right) + h^2 \left( c_1 k_1^{(2)} + c_2 k_2^{(2)} \right), \tag{2.15}
\]
where
\[
y_0 = \alpha, \quad k_1^{(1)} = f(y_0), \quad k_2^{(1)} = f(y_0 + h a_{21} k_1^{(1)}),
\]
Specific nonzero constants, in the extended Runge-Kutta-like formulae of order 3 (ERK3) (Wu and Xia, 2006) are
\[
b_1 = 1, \quad b_{21} = \frac{1}{3}, \quad c_2 = \frac{1}{2}
\]
The specific formula of interest is
\[
y_{n+1} = y_n + h f_n + \frac{1}{3} h^2 f' \left( y_n + \frac{1}{3} h f_n \right),
\]
or
\[
y_{n+1} = y_n + h k_1^{(1)} + \frac{1}{2} h^2 k_2^{(2)},
\]
with the local truncation error
\[
T(t, h) = \frac{1}{72} h^4 \left[ f_{yy} f_n^3 + 6 f_y f_{yy} f_n^2 + 3 f_y^3 f_n \right] + O(h^5),
\]
where \( f_n = f(y_n), \) \( f'_n = f'_y(y_n) f(y_n) \). By using Eq. (2.17), the following result yields.

Lemma 2.2: Let \( f(y) \) belong to \( C^3[a, b] \) and let its derivatives be bounded. Also, assume that there exists \( P, M, \) positive numbers, such that \( |f(y)| < M, \quad \left| \frac{df}{dy} \right| < \frac{p}{M^j-1}, \quad j \leq 3 \) then
\[
y(t_n + h) - y_{n+1} = \frac{1}{72} h^4 \left[ f_{yy} f_n^3 + 6 f_y f_{yy} f_n^2 + 3 f_y^3 f_n \right] + O(h^5),
\]
and
\[
|y(t_n + h) - y_{n+1}| \leq \frac{5}{36} h^4 M P^3 + O(h^5).
\]

A fuzzy Cauchy problem

Consider the autonomous fuzzy initial value problem
\[
y'(t) = f(y(t)), \quad y(0) = y_0, \quad t \in I = [0, T],
\]
where \( f \) is a continuous mapping from \( E^1 \) into \( E^1 \) and \( y_0 \in E^1 \) with \( r \)-level sets
\[
[y_0]_r = [y(t); r], \quad r \in (0, 1].
\]
The extension principle of Zadeh leads to the following definition of \( f(y(t)) \), when \( y = y(t) \) is a fuzzy number:
\[
f(y(t)) (s) = \sup \{ y(r) | s = f(r), \quad s \in \mathbb{R} \}.
\]
From this, it follows that
\[
[f(y(t))](r) = [f_1(y; r), \quad f_2(y; r)], \quad r \in (0, 1],
\]
where
\[
f_1(y; r) = \min \{ f(u) | u \in [y_1(t; r), y_2(t; r)] \},
\]
\[
f_2(y; r) = \max \{ f(u) | u \in [y_1(t; r), y_2(t; r)] \}.
\]
The mapping \( f(y) \) is a fuzzy function and the derivative \( f'(y) \) is defined by
\[
\left[ f'(y(t)) \right]_r = [f'_1(y; r), f'_2(y; r)], \quad t \in I, \quad r \in (0, 1],
\]
provided that this equation determines the fuzzy number \( f'(y) \in E^1 \), where
\[
f'_1(y; r) = \min \{ f'(u) | u \in [y_1(t; r), y_2(t; r)] \}, \quad f'_2(y; r) = \max \{ f'(u) | u \in [y_1(t; r), y_2(t; r)] \}.
\]
Sufficient conditions for the existence of a unique solution to Eq. (3.1) is that \( f \) satisfies the Lipschitz condition
\[
\| f(y) - f(z) \| \leq L \| y - z \|, \quad L > 0.
\]
By theorem 5.2 in (Kaleva, 1987), we may replace Eq. (3.1) by the equivalent system
\[
\begin{align*}
y'(t; r) &= f(y(t); r) = f_1(y; r) = F \left( y(t; r), \quad \tilde{y}(t; r) \right), \quad \tilde{y}(0; r) = \tilde{y}_0(r), \\
\bar{y}'(t; r) &= \bar{f}(y(t); r) = f_2(y; r) = G \left( y(t; r), \quad \tilde{y}(t; r) \right), \quad \tilde{y}(0; r) = \tilde{y}_0(r),
\end{align*}
\]
for \( r \in (0, 1] \).

The third-order fuzzy Runge-Kutta-like formula

We consider fuzzy Cauchy problem (3.1) with step-size \( h = T/N \) and grid points
\[
t_j = jh, \quad j = 0, 1, 2, ..., N.
\]
Let the exact solution \( \left[ Y(t) \right]_r = [Y_1(t; r), \quad Y_2(t; r)] \) be approximated by some
\( \left[ y(t) \right]_r = [y_1(t; r), \quad y_2(t; r)] \). From (2.16), we define
\[
y_1(t_{n+1}; r) - y_1(t_n; r) = h k_1^{(1)} (y(t_n; r)) + \frac{1}{2} h^2 k_2^{(2)} (y(t_n; r)),
\]
where
\[
k_1^{(1)} = f' (y_n + \frac{1}{3} h f_n),
\]
\[
k_2^{(2)} = f' \left( y_n + \frac{1}{3} h f_n \right) + h^2 f'' \left( y_n + \frac{1}{3} h f_n \right).
\]
\[ y_2(t_{n+1}; r) - y_2(t_n; r) = h k_{1,1}^{(1)}(y(t_n; r)) + \frac{1}{2} h^2 k_{2,2}^{(2)}(y(t_n; r)), \]  
\tag{4.2}
\]

where
\[ [k_{1,1}^{(1)}(y(t; r))]_r = [k_{1,1}^{(2)}(y(t; r)), k_{1,1}^{(1)}(y(t; r))], \]
\[ [k_{2,2}^{(2)}(y(t; r))]_r = [k_{2,2}^{(1)}(y(t; r)), k_{2,2}^{(2)}(y(t; r))]. \]  
\tag{4.3}

and
\[ k_{1,1}^{(1)}(y(t; r)) = \min \{ f(u) \mid u \in [y_1(t; r), y_2(t; r)] \}, \]
\[ k_{1,1}^{(1)}(y(t; r)) = \max \{ f(u) \mid u \in [y_1(t; r), y_2(t; r)] \}, \]

and
\[ k_{2,2}^{(2)}(y(t; r)) = \min \{ f'(w) \mid w \in [z_1(y(t; r), z_2(y(t; r))] \}, \]
\[ k_{2,2}^{(2)}(y(t; r)) = \max \{ f'(w) \mid w \in [z_1(y(t; r), z_2(y(t; r))] \}. \]  
\tag{4.4}

where
\[ z_1(y(t; r)) = y_1(t; r) + \frac{1}{3} h k_{1,1}^{(1)}(y(t; r)), \]
\[ z_2(y(t; r)) = y_2(t; r) + \frac{1}{3} h k_{1,2}^{(1)}(y(t; r)). \]  
\tag{4.5}

Define
\[ F[y(t; r)] = h k_{1,1}^{(1)}(y(t; r)) + \frac{1}{2} h^2 k_{2,2}^{(2)}(y(t; r)), \]
\[ G[y(t; r)] = h k_{1,2}^{(1)}(y(t; r)) + \frac{1}{2} h^2 k_{2,2}^{(2)}(y(t; r)). \]  
\tag{4.6}

The exact and approximated solutions at \( t_n \), \( 0 \leq n \leq N \) are denoted by:
\[ [y(t_n)] = [y_1(t_n; r), y_2(t_n; r)], \]
\[ [y(t_n)] = [y_1(t_n; r), y_2(t_n; r), y_3(t_n; r)], \]

respectively. The solution is calculated by grid points at (4.1). From (4.2) and (4.6), we have
\[ y_1(t_{n+1}; r) = y_1(t_n; r) + F[y(t_n; r)], \]
\[ y_2(t_{n+1}; r) = y_2(t_n; r) + G[y(t_n; r)]. \]  
\tag{4.7}

Also
\[ Y_1(t_{n+1}; r) \approx Y_1(t_n; r) + F[y(t_n; r)], \]
\[ Y_2(t_{n+1}; r) \approx Y_2(t_n; r) + G[y(t_n; r)]. \]  
\tag{4.8}

The following lemmas (Ma et al. 1999) will be applied to show the convergence of these approximates
\[ \lim_{h \to 0} y_1(t_n; r) = Y_1(t_n; r), \]
\[ \lim_{h \to 0} y_2(t_n; r) = Y_2(t_n; r). \]

Lemma 4.1. Let the sequence of numbers \( \{ W_n \}_{n=0}^N \) satisfy
\[ |W_{n+1}| \leq A |W_n| + B, \quad 0 \leq n \leq N - 1, \]
for some given positive constants \( A \) and \( B \). Then,
\[ |W_n| \leq A^n |W_0| + B \frac{A^{n-1}}{A-1}, \quad 0 \leq n \leq N. \]

Lemma 4.2. Let the sequence of numbers \( \{ V_n \}_{n=0}^N \) satisfy
\[ |V_{n+1}| \leq |V_n| + A \max \{ |W_n|, |V_n| \} + B, \]
\[ |V_{n+1}| \leq |V_n| + A \max \{ |W_n|, |V_n| \} + B, \]
for some given positive constants \( A \) and \( B \), and denote
\[ U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N. \]

Then,
\[ |U_n| \leq A^n |U_0| + B \frac{A^{n-1}}{A-1}, \quad 0 \leq n \leq N, \]
where
\[ A = 1 + 2A \quad \text{and} \quad B = 2B. \]

Let \( F(u, v) \) and \( G(u, v) \) be obtained by substituting \( [y(t)]_r = [u, v] \) in (4.6),
\[ F(u, v) = h k_{1,1}^{(1)}(u, v) + \frac{1}{2} h^2 k_{2,2}^{(2)}(u, v), \]
\[ G(u, v) = h k_{1,2}^{(1)}(u, v) + \frac{1}{2} h^2 k_{2,2}^{(2)}(u, v). \]  
\tag{4.9}

The domain of \( F \) and \( G \) is
\[ K = \{ (u, v) \mid -\infty < v < \infty, -\infty < u < v \}, \quad 0 \leq t \leq T, \]
Theorem 4.3. Let \( F(u, v) \) and \( G(u, v) \) belong to \( C^3(K) \) and let the partial derivative of \( F \) and \( G \) be bounded over \( K \). Then, for arbitrary fixed, \( r, 0 \leq r \leq 1 \), the approximate solutions (4.7) converge to the exact solutions \( Y_i(t_n; r) \) and \( Y_2(t_n; r) \) uniformly in \( t \).

**Proof.** The proof is similar to Theorem 4.3 in (Ghazanfari et al. 2011).

Next we consider ‘derivative-free’ patterns corresponding to the formula (2.15). For this purpose it is enough to approximate \( f' \) to some given accuracy by employing the current and previous evaluations of \( f \), such as

\[
k_1^{(2)} = f'(y_n) \approx \frac{f(y_{n+1}) - f(y_n)}{h},
\]

\[
k_2^{(2)} = f'(y_n + h \frac{b_1 + c_1}{2} f(y_n) + \frac{b_1}{2} f(y_n + h a_{21} f(y_n)) + c_2 f(y_n + h b_{21} f(y_n)) - c_1 f(y_{n-1})}
\]

This motivates a class of two-step Runge-Kutta formulae as follows

\[
y_{n+1} = y_n + h \left[ f(y_n) + \frac{1}{2} f\left(y_n + \frac{5}{6} f(y_n) - \frac{1}{2} f\left(y_n - h \frac{5}{6} f(y_n)\right)\right]\right],
\]

which is a two-step Runge-Kutta method of third order and only needs two function evaluations of \( f \) per step.

The fuzzy Euler, Runge-Kutta and two-step extended Runge-Kutta methods

Consider fuzzy initial value problem (Ma et al. 1999), (Abbasbandy et al. 2004)

\[
y'(t) = f(t, y(t)),
y(0) = y_0, \quad t \in [0, T],
\]

where \( \hat{\cdot} \) denotes Seikkala differentiation, \( 0 = t_0 < t_1 < \cdots < N = T, f \in C[\mathbb{R}^+ \times E^1, E^1] \) and \( y_0 \in E^1 \) with r-level sets

\[
y_0^r = \{ y(t_0; r), y_2(t_0; r) \}, \quad r \in (0, 1).
\]

The extension principle of Zadeh leads to the following definition of \( f(t, y(t)) \), when \( y = y(t) \) is a fuzzy number:

\[
f(t, y(t)) (s) = \sup \{ y(r) | s = f(t, r) \}, s \in \mathbb{R}.
\]

From this, it follows that

\[
[ f(t, y(t))]_r = [ f_1(t, y; r), f_2(t, y; r)], \quad r \in (0, 1),
\]

where

\[
f_1(t, y; r) = \min \{ f(t, u) | u \in [ y_1(t; r), y_2(t; r) ] \},
f_2(t, y; r) = \max \{ f(t, u) | u \in [ y_1(t; r), y_2(t; r) ] \}.
\]

We may replace Eq. (5.1) by the equivalent system [12]

\[
y'(t; r) = \hat{f}(t, y(t)) = f_1(t, y_1(t; r), y_2(t; r)),
\]

\[
y'(t; r) = \hat{y}(t, y(t)) = f_2(t, y_1(t; r), y_2(t; r)),
\]

for \( r \in (0, 1) \). The exact solution \( Y(t) \) is approximated by some \( y(t)_r = [ y_1(t; r), y_2(t; r) ] \). The exact and approximate solutions at \( t_n, 0 \leq n \leq N \), are denoted by \( [ Y(t_n)]_r = [ y_1(t_n; r), y_2(t_n; r) ] \) and \( [ y(t_n)]_r = [ y_1(t_n; r), y_2(t_n; r) ] \), respectively.

**Euler** method (Ma et al. 1999) is based on the first-order approximation of \( Y'_1(t; r), Y'_2(t; r) \) and denoted by

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + h f_1(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + h f_2(t_n, y_1(t_n; r), y_2(t_n; r)),
\]

for more information, see (Ma et al. 1999).

**The Runge-Kutta method of order 2 (RK2)** is based on the 2nd-order approximation of \( Y'_1(t; r), Y'_2(t; r) \) and denoted by

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{2} F(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{2} G(t_n, y_1(t_n; r), y_2(t_n; r)),
\]

or

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{2} F(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{2} G(t_n, y_1(t_n; r), y_2(t_n; r)),
\]

\[
 k_1 = \left[ k_{1,1}, k_{1,2} \right], \quad k_2 = \left[ k_{2,1}, k_{2,2} \right],
\]

or

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{2} F(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{2} G(t_n, y_1(t_n; r), y_2(t_n; r)),
\]

\[
 k_1 = \left[ k_{1,1}, k_{1,2} \right], \quad k_2 = \left[ k_{2,1}, k_{2,2} \right],
\]

or

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{2} F(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{2} G(t_n, y_1(t_n; r), y_2(t_n; r)),
\]

\[
 k_1 = \left[ k_{1,1}, k_{1,2} \right], \quad k_2 = \left[ k_{2,1}, k_{2,2} \right],
\]

or

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{2} F(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{2} G(t_n, y_1(t_n; r), y_2(t_n; r))
\]

\[
 k_1 = \left[ k_{1,1}, k_{1,2} \right], \quad k_2 = \left[ k_{2,1}, k_{2,2} \right],
\]

or

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{2} F(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{2} G(t_n, y_1(t_n; r), y_2(t_n; r))
\]

\[
 k_1 = \left[ k_{1,1}, k_{1,2} \right], \quad k_2 = \left[ k_{2,1}, k_{2,2} \right],
\]

or

\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{2} F(t_n, y_1(t_n; r), y_2(t_n; r)),
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{2} G(t_n, y_1(t_n; r), y_2(t_n; r))
\]

\[
 k_1 = \left[ k_{1,1}, k_{1,2} \right], \quad k_2 = \left[ k_{2,1}, k_{2,2} \right],
\]
where
\[ F(t, y_1(t_n; r)) = k_{1,1}(t, y_1(t_n; r)) + k_{2,1}(t, y_1(t_n; r)), \]
\[ G(t, y_1(t_n; r)) = k_{1,2}(t, y_1(t_n; r)) + k_{2,2}(t, y_1(t_n; r)), \]
\[ k_{1,1}(t, y(t; r)) = \min \{ h f(t, u) \mid u \in [y_1(t; r), y_2(t; r)] \}, \]
\[ k_{1,2}(t, y(t; r)) = \max \{ h f(t, u) \mid u \in [y_1(t; r), y_2(t; r)] \}, \]
and
\[ k_{2,1}(t, y(t; r)) = \min \{ h f(t + h, w) \mid w \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))] \}, \]
\[ k_{2,2}(t, y(t; r)) = \max \{ h f(t + h, w) \mid w \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))] \} \] (5.6)
and
\[ z_{1,1}(t, y(t; r)) = y_1(t; r) + k_{1,1}(t, y(t; r)), \]
\[ z_{1,2}(t, y(t; r)) = y_2(t; r) + k_{1,2}(t, y(t; r)). \]
For more information, see (Abbasbandy et al. 2004).

Now, TSERKO3 is based on the 3nd-order approximation of
\[ ™wŒ
X
™wŒ
± \& \$U]\$UŒ],
\[ ™wŒ
X
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± \& \$U]\$UŒ, \]
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± \& \$U]\$UŒ, \]
where
\[ k_{1,1}^{(1)}(y(t; r)) = \min \{ f(u) \mid u \in [y_1(t; r), y_1(t; r)] \}, \]
\[ k_{1,2}^{(1)}(y(t; r)) = \max \{ f(u) \mid u \in [y_1(t; r), y_1(t; r)] \}, \]
\[ k_{1,1}^{(2)}(y(t; r)) = \min \{ f(w) \mid w \in [z_{2,1}(y(t; r)), z_{2,2}(y(t; r))] \}, \]
\[ k_{1,2}^{(2)}(y(t; r)) = \max \{ f(w) \mid w \in [z_{2,1}(y(t; r)), z_{2,2}(y(t; r))] \}, \]
and
\[ z_{2,1}(y(t; r)) = y_1(t; r) + \frac{5}{6} h k_{1,1}^{(1)}(y(t; r)), \]
\[ z_{2,2}(y(t; r)) = y_2(t; r) + \frac{5}{6} h k_{1,2}^{(1)}(y(t; r)). \]

**Example**

Example 1. Consider the fuzzy initial value problem (Ma et al. 1999).
\[ y'(t) = y(t), \quad t \in I = [0, T], \]
\[ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), \quad r \in (0, 1). \] (6.1)
The exact solution is given by \( Y_1(t, r) = y_1(0; r) e^t, Y_2(t, r) = y_2(0; r) e^t \), hence
\[ Y(t; r) = [Y_1(t, r) = y_1(0; r) e^t, Y_2(t, r) = y_2(0; r) e^t]. \]
Also, \( Y(1; r) \) is given by
\[ Y(1; r) = [(0.75 + 0.25r) e^1, (1.125 - 0.125r) e^1], \quad r \in (0, 1). \]
Using the fuzzy TSERKO3, we have
\[ y_{n+1} = y_n + \frac{1}{2} f(y_n) + \frac{1}{2} f \left( y_n + \frac{5}{6} f(y_n) \right) - \frac{1}{2} f \left( y_{n-1} + \frac{5}{6} f(y_{n-1}) \right). \]
Hence,
\[ y_1(t_{n+1}; r) = y_1(t_n; r) \left[ 1 + \frac{3}{2} h + \frac{5}{12} h^2 \right] - y_2(t_{n-1}; r) \left[ \frac{1}{2} h + \frac{5}{12} h^2 \right], \]
\[ y_2(t_{n+1}; r) = y_2(t_n; r) \left[ 1 + \frac{3}{2} h + \frac{5}{12} h^2 \right] - y_1(t_{n-1}; r) \left[ \frac{1}{2} h + \frac{5}{12} h^2 \right]. \]
The results of ERK3, TSERKO3, RK2 and Euler method with \( h = 0.1 \) at \( t = 1 \) are shown in Table 1.
The exact and approximate solutions are compared and plotted at $t = 1$ in Fig. 1.

### Table 1. The results of ERK3, TSERK03, RK2 and Euler method with $h = 0.1$ at $t = 1$.  

| $r$ | ERK3 | RK2 | TSERK | Euler | Exact | $|y_1 - Y_1|$ | $|y_{rk_1} - Y_1|$ | $|y_{ts_1} - Y_1|$ | $|E_1 - Y_1|$ |
|-----|------|-----|-------|-------|-------|-------------|----------------|----------------|----------------|
| 0.0 | 2.038633 | 2.035561 | 2.038800 | 1.945307 | 2.038711 | 7.842448e-5 | 3.150736e-3 | 8.062866e-5 | 1.127600e-0 |
| 0.1 | 2.106587 | 2.103413 | 2.106639 | 2.010150 | 2.106688 | 8.103863e-5 | 3.255761e-3 | 2.941706e-5 | 1.019338e-0 |
| 0.2 | 2.174542 | 2.171265 | 2.174579 | 2.074994 | 2.174825 | 8.365278e-5 | 3.360785e-3 | 4.646277e-5 | 9.151160e-1 |
| 0.3 | 2.242496 | 2.239117 | 2.242607 | 2.139832 | 2.242583 | 8.626693e-5 | 3.465810e-3 | 2.449152e-5 | 8.162940e-1 |
| 0.4 | 2.309651 | 2.306969 | 2.309544 | 2.204681 | 2.309540 | 8.888108e-5 | 3.570835e-3 | 4.458104e-5 | 7.174719e-1 |
| 0.5 | 2.378405 | 2.374821 | 2.378551 | 2.269525 | 2.378497 | 9.149523e-5 | 3.675859e-3 | 5.440010e-5 | 6.186498e-1 |
| 0.6 | 2.446360 | 2.442673 | 2.446482 | 2.334368 | 2.446454 | 9.410938e-5 | 3.780884e-3 | 2.835439e-5 | 5.192727e-1 |
| 0.7 | 2.514314 | 2.510525 | 2.514364 | 2.399212 | 2.514411 | 9.672353e-5 | 3.885908e-3 | 4.669132e-5 | 4.210056e-1 |
| 0.8 | 2.582268 | 2.578377 | 2.582465 | 2.464055 | 2.582368 | 9.933768e-5 | 3.990933e-3 | 7.262966e-5 | 3.221836e-1 |
| 0.9 | 2.650223 | 2.646229 | 2.650339 | 2.528899 | 2.650325 | 1.019518e-4 | 4.095957e-3 | 1.421725e-5 | 2.233615e-1 |
| 1.0 | 2.718177 | 2.714081 | 2.718226 | 2.593742 | 2.718282 | 1.045660e-4 | 4.200982e-3 | 5.582846e-5 | 1.245394e-1 |

\[

t = 0.1
\]

![Figure 1. $h = 0.1$](image-url)

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Example 2. Consider the fuzzy initial value problem (Palligkinis et al. 2009),
\[ y'(t) = f(y(t)) = cy(t), \quad t \in I = [0,1], \]
\[ y(0) = y_0, \quad y_0 = (8/8.5/9), \quad c = (1/2/3). \]
The analytical form is
\[ Y_1(t; r) = y_1(0; r)e^{c_1 t} = (8 + 0.5r)e^{(1+r)t}, \]
\[ Y_2(t; r) = y_2(0; r)e^{c_2 t} = (9 - 0.5r)e^{(3-r)t}. \]
With the initial condition \((y_1(0), y_2(0))\). Also,
\[ y_{n+1} = y_n + h f_n + \frac{1}{2} h^2 f'(y_n + \frac{1}{3} h f_n). \]

Using the ERK3 formula, we have
\[ y_1(t_{n+1}; r) = y_1(t_n; r) + h f_1(y(t_n; r)) + \frac{1}{2} h^2 f_1(y(t_n; r) + \frac{1}{3} h f(y(t_n; r))), \quad (6.7) \]
\[ y_2(t_{n+1}; r) = y_2(t_n; r) + h f_2(y(t_n; r)) + \frac{1}{2} h^2 f_2(y(t_n; r) + \frac{1}{3} h f(y(t_n; r))), \quad (6.8) \]
where
\[ f_1(y(t_n; r)) = \min \{ b.u \mid u \in [y_1(t_n; r), y_2(t_n; r)], \quad b \in [c_1(r), c_2(r)] \} = c_1(r) y_1(t_n; r), \]
\[ f_2(y(t_n; r)) = \max \{ b.u \mid u \in [y_1(t_n; r), y_2(t_n; r)], \quad b \in [c_1(r), c_2(r)] \} = c_2(r) y_2(t_n; r). \]

And
\[ f_1'(w(t_n; r)) = \min \{ b^2.s \mid s \in [z_{1.1}(t_n; r), z_{1.2}(t_n; r)], \quad b \in [c_1(r), c_2(r)] \} \]
\[ = c_1^2(r) y_1(t_n; r) + \frac{1}{3} h f_1(y(t_n; r)) = c_1^2(r) y_1(t_n; r) + \frac{1}{3} h c_1(r) y_1(t_n; r) \]
\[ = c_1^2(r) y_1(t_n; r) + \frac{1}{3} h c_1(r) y_1(t_n; r) \]
\[ f_2'(w(t_n; r)) = \max \{ b^2.s \mid s \in [z_{1.1}(t_n; r), z_{1.2}(t_n; r)], \quad b \in [c_1(r), c_2(r)] \} \]
\[ = c_2^2(r) y_2(t_n; r) + \frac{1}{3} h f_2(y(t_n; r)) = c_2^2(r) y_2(t_n; r) + \frac{1}{3} h c_2(r) y_2(t_n; r) \]
\[ = c_2^2(r) y_2(t_n; r) + \frac{1}{3} h c_2(r) y_2(t_n; r) \]

such that
\[ z_{1.1}(t_n; r) = y_1(t_n; r) + \frac{1}{3} h f_1(y(t_n; r)) = y_1(t_n; r) + \frac{1}{3} h c_1(r) y_1(t_n; r) = y_1(t_n; r) \left[ 1 + \frac{1}{3} h c_1(r) \right], \]
\[ z_{1.2}(t_n; r) = y_2(t_n; r) + \frac{1}{3} h f_2(y(t_n; r)) = y_2(t_n; r) + \frac{1}{3} h c_2(r) y_2(t_n; r) = y_2(t_n; r) \left[ 1 + \frac{1}{3} h c_2(r) \right]. \]

Hence,
\[ y_1(t_{n+1}; r) = y_1(t_n; r) \left[ 1 + \frac{1}{3} h c_1(r) \right], \quad y_2(t_{n+1}; r) = y_2(t_n; r) \left[ 1 + \frac{1}{3} h c_2(r) \right]. \]
The results of ERK3, TSERKO3, RK2 and Euler method with \( h = 0.1 \) at \( t = 1 \) are shown in Table 2. The exact and approximate solutions are compared and plotted at \( t = 1 \) in Fig. 2.

CONCLUSION

Two-step Extended Runge-Kutta-like formulae of order three has been applied in this paper. This procedure consists of using an extended Runge-Kutta method with local truncation error of order 4. A clear advantage to this technique is that only evaluations of \( f \) instead of \( f \) and \( f' \) per step. This method can be used for those fuzzy initial value problems that \( f' \) may not be exists.
### Table 2. The results of ERK3, TSERKO3, RK2 and Euler method with $h = 0.1$ at $t = 1$. 

| $r$ | $y_1(t_0)$ | $y_{rk1}(t_0)$ | $y_{tserk1}$ | $E_1$ | $|y_{rk1} - Y_1|$ | $|y_{tserk1} - Y_1|$ | $|E_1 - Y_1|$ | $|y - Y_1|$ |
|-----|-------------|----------------|--------------|-------|-----------------|-----------------|-----------------|-------------|
| 0.0 | 2.1745e+1  | 2.1713e+1     | 2.1743e+1    | 2.0750e+1 | 2.1748e-4       | 3.3602e-3       | 3.0025e-3       | 9.9631e-1   |
| 0.1 | 2.4182e+1  | 2.4134e+1     | 2.4179e+1    | 2.2857e+1 | 2.4184e-2       | 4.9365e-3       | 4.8463e-3       | 1.3262e+0   |
| 0.2 | 2.6891e+1  | 2.6885e+1     | 2.6517e+1    | 2.6893e+1 | 2.1112e-3       | 7.0721e-2       | 7.5660e-3       | 1.7356e+0   |
| 0.3 | 2.9902e+1  | 2.9822e+1     | 2.9693e+1    | 2.9905e+1 | 2.9078e-3       | 9.9212e-2       | 1.1486e-2       | 2.2390e+0   |
| 0.4 | 3.3248e+1  | 2.9806e+1     | 3.3236e+1    | 3.3253e+1 | 4.7956e-3       | 1.3671e-1       | 1.7025e-1       | 2.8534e+0   |
| 0.5 | 3.6967e+1  | 3.3116e+1     | 3.6949e+1    | 3.3376e+1 | 6.9188e-3       | 1.8551e-1       | 2.4723e-1       | 3.5981e+0   |
| 0.6 | 4.1100e+1  | 3.6788e+1     | 4.1075e+1    | 4.1110e+1 | 9.8794e-3       | 2.4835e-1       | 3.5262e-1       | 4.4953e+0   |
| 0.7 | 4.5694e+1  | 4.0862e+1     | 4.5658e+1    | 4.0137e+1 | 1.3887e-2       | 3.2858e-1       | 4.9506e-1       | 5.5704e+0   |
| 0.8 | 5.0798e+1  | 4.5379e+1     | 5.0748e+1    | 4.3964e+1 | 1.9252e-1       | 4.3018e-1       | 6.8540e-1       | 8.6525e+0   |
| 0.9 | 5.6469e+1  | 5.0387e+1     | 5.6402e+1    | 4.8120e+1 | 2.6359e-2       | 5.5795e-1       | 9.3713e-1       | 8.3757e+0   |
| 1.0 | 6.2771e+1  | 5.5938e+1     | 6.2680e+1    | 5.2630e+1 | 3.5691e-1       | 7.1761e-1       | 1.2670e-1       | 1.0177e+1   |

![Graph](image-url)
REFERENCES


